The metric on $C(G, \mathbb{C})$

Defn: If G is an open set in $\mathbb{C}$ and $\mathbb{C}$ is equipped with standard metric $d(z,w) = |z-w|$.
we denote
$$C(G,\mathbb{C}) = \{\text{all continuous function } f: G \to \mathbb{C}\}.$$

Q: How to define a metric on $C(G,\mathbb{C})$?

Recall: A metric d on a set X: $\forall x, y, z$

1. $d(x,y) \geq 0$
2. $d(x,y) = 0 \iff x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,z) \leq d(x,y) + d(y,z)$.

Naive idea:

define $d(f,g) = \sup\{|f(z) - g(z)| : z \in G\}$
for $f, g \in C(G,\mathbb{C})$.

But $d(f,g)$ may be ∞ in this way

A better way:

Defn: exhaustion sequence of G.

A sequence of compact subset $\{K_n\}_{n=1}^{\infty}$ of G is called an exhaustion sequence of compact set for G

is called an exhaustion sequence of compact set for G if

(1) $K_1 \subset K_2^\circ \subset K_2 \subset K_3^\circ \subset K_3 \subset \cdots$

(2) Every compact subset K of G is contained in some K_n. In particular,

$$\bigcup_{n=1}^{\infty} K_n = G$$

Existence of Exhaution sequence

proposition 1.2: Let $G \subset \mathbb{C}$ be open. Set

$$K_n = \{z \in G : d(z, \mathbb{C}-G) \geq \tfrac{1}{n}\} \cap \{z : |z| \leq n\}$$

Then (1) (2) holds.

Moreover,

(3) Every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$.

Pf: P143 in the book.

Now for each K_n, we can define a metric on $C(K_n, \mathbb{C})$:

$$\rho_n(f,g) = \sup\{d(f(z), g(z)) : z \in K_n\}$$

for $f, g \in C(K_n, \mathbb{C})$.

Q: How to use this to define a metric on $C(G, \mathbb{C})$.

Observation: $d_n(f,g) \triangleq \dfrac{\rho_n(f,g)}{1+\rho_n(f,g)}$ is also a metric on $C(K_n, \mathbb{C})$.

Pf: ① $d_n(f,g) \geq 0$

② $d_n(f,g) = 0 \Leftrightarrow \rho_n(f,g) = 0 \Leftrightarrow f = g$

③ $d_n(f,g) = d_n(g,f)$

④ Triangle inequality.

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)} \leq \frac{\rho_n(f,h)}{1+\rho_n(f,h)} + \frac{\rho_n(h,g)}{1+\rho_n(h,g)}$$

We write $a = \rho_n(f,g)$, $b = \rho_n(f,h)$, $c = \rho_n(h,g)$

Since ρ_n is a metric $\Rightarrow a \leq b + c$

Now set $\varphi(x) = \frac{x}{1+x}$, we need to show

$$\varphi(a) \leq \varphi(b) + \varphi(c).$$

Note $\varphi(x) = 1 - \frac{1}{1+x}$, $\varphi'(x) = \frac{1}{(1+x)^2}$, $\varphi''(x) = -2\frac{1}{(1+x)^3}$.

$\Rightarrow \varphi \nearrow$ $\varphi' \searrow$ on $x \in [0, +\infty)$.

Then $\varphi(a) \leq \varphi(b+c)$.

We fix $c > 0$ and set $\psi(b) = \varphi(b) + \varphi(c) - \varphi(b+c)$

Note $\begin{cases} \psi(0) = 0 \\ \psi'(b) = \varphi'(b) - \varphi'(b+c) \geq 0 \text{ as } \varphi' \searrow \end{cases}$

$\Rightarrow \psi(b) \geq 0$ for all $b \geq 0$. Thus

$\varphi(b) + \varphi(c) \geq \varphi(b+c)$

$\Rightarrow \varphi(a) \leq \varphi(b) + \varphi(c)$.

Metric on $C(G, \mathbb{C})$:

Define for $f, g \in C(G, \mathbb{C})$

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) : z \in K_n\}$$

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1+\rho_n(f, g)}$$

Then

proposition 1.6: $(C(G, \mathbb{C}), \rho)$ is a metric space.

pf: page 144 in the book

proposition 1.10:

A sequence $f_n \to f$ in $(C(G, \mathbb{C}), \rho)$ (i.e $\rho(f_n, f) \to 0$) $\Longleftrightarrow$ f_n converges to f on every compact subset of G.

Idea: when $\rho(f_j, f) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f_j, f)}{1+\rho_n(f_j, f)} \to 0$

on every K_n, $\frac{\rho_n(f_j, f)}{1+\rho_n(f_j, f)} \to 0 \Rightarrow \rho_n(f_j, f) \to 0$

proposition 1.12: $C(G, \mathbb{C})$ is a complete metric space under ρ.

Recall: "complete" means every Cauchy sequence $\{f_j\}$ has a limit f in $C(G, \mathbb{C})$.

Idea of pf: If $\{f_j\}$ is Cauchy in $C(G, \mathbb{C})$, then

$$\rho_n(f_j, f_k) \to 0 \quad \text{as } j, k \to \infty.$$

That is, $\{f_j\}$ is Cauchy on every K_n

That is, $\{f_j\}$ is Cauchy on every K_n

Thus $\{f_j\}$ has a pointwise limit f on K_n.

Moreover, $f_j \to f$ on K uniformly.

Since the uniform limit of continuous functions is continuous. Thus $f \in C(G, \mathbb{C})$.

Defn. A set $\mathcal{F} \subset C(G, \mathbb{C})$ is normal if each sequence in $\mathcal{F}$ has a subsequence which converges to a function f in $C(G, \mathbb{C})$.

proposition 1.15. A set $\mathcal{F} \subset C(G, \mathbb{C})$ is normal iff its closure is compact.

proposition 1.16. A set $\mathcal{F} \subset C(G, \mathbb{C})$ is normal iff for every compact set $K \subset G$ and $\delta > 0$ there are functions $f_1, \cdots, f_n$ in $\mathcal{F}$ such that for $f \in \mathcal{F}$, there $\exists$ some $1 \le k \le n$ s.t

$$\sup\{d(f(z), f_k(z)) : z \in K\} < \delta.$$

Idea of Pf: Use the compactness of $\overline{\mathcal{F}}$.

Defn: ① A set $\mathcal{F} \subset C(G, \mathbb{C})$ is equicontinuous at a point $z_0 \in G$ if for every $\varepsilon > 0$, there $\exists$ $\delta > 0$ s.t for $|z - z_0| < \delta$,

$$d(f(z), f(z_0)) < \varepsilon.$$

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for every $f \in \mathcal{F}$

② $\mathcal{F}$ is equicontinuous over a set $E \subset G$ if for every $\varepsilon > 0$ there $\exists\, \delta > 0$ s.t for z and z' in E and $|z - z'| < \delta$,

$$d(f(z), f(z')) < \varepsilon.$$

for all $f \in \mathcal{F}$.

Proposition 1.22. Suppose $\mathcal{F} \subset C(G, \mathbb{C})$ is equicontinuous at every pt of G then $\mathcal{F}$ is equicontinuous over every compact subset of G.

Arzela–Ascoli Thm: A set $\mathcal{F} \subset C(G, \mathbb{C})$ is normal iff the following two conditions are satisfied:

(a) for each $z \in G$, $\exists\, M > 0$ s.t

$$|f(z)| \le M \text{ for all } f \in \mathcal{F}.$$

(b) $\mathcal{F}$ is equicontinuous at each pt of G

($\Leftrightarrow$ $\mathcal{F}$ is equicontinuous over every compact subset $K \subset G$)

Pf: We will only prove the "$\Leftarrow$" direction. Suppose $\mathcal{F}$ satisfies (a) and (b). We will show $\mathcal{F}$ is normal.

Let $\{z_n\}_{n=1}^{\infty}$ be the sequence of all pts in G with rational real and imaginary parts. Then rational real and imaginary parts. Then

$\{z_n\}$ is dense in G. Let $\{f_k\}$ be a sequence in $\mathcal{F}$.

We find a subsequence $f^{(1)}$ of $\{f_k\}$

$$f_j^{(1)}: f_{k_1}, f_{k_2}, f_{k_3} \cdots f_{k_j} \cdots \text{ which converges}$$

at z_1.

Then we find a subsequence $f_j^{(2)}$ of $\{f_j^{(1)}\}$ s.t

$f_j^{(2)}$ converges at z_2.

Inductively, we find a subsequence $f_j^{(l)}$ of $\{f_j^{(l-1)}\}$ s.t $f_j^{(l)}$ converges at z_l. Then by diagonalization method, we can find a subsequence $\{f_l^{(l)}\}$ of $\{f_k\}$ s.t

$$f_1^{(1)}, f_2^{(2)}, \cdots \text{ converges at every } z_n.$$

We will still call this subsequence $\{f_k\}$.

Next we prove.

Lemma: Fix a compact subset $K \subset G$.

Let $\varepsilon > 0$. $\exists J > 0$ s.t

$$k, j \geq J \Rightarrow \sup\{d(f_k(z), f_j(z)) : z \in K\} < \varepsilon.$$

Pf. write $R = d(K, \mathbb{C} - G) > 0$

Let $K_1 = \{z \in G : d(z, K) \leq \frac{R}{2}\}$

Then K_1 is compact and

$$K \subset K_1^{\circ} \subset K_1 \subset G.$$

By assumption (b), $\mathcal{F}$ is equicontinuous on K_1.

Then $\exists\ \delta$ with $0 < \delta < \frac{R}{\ }$ s.t

By assumption (b), $\mathcal{F}$ is equicontinuous on K_1.
Thus $\exists\ \delta > 0$ with $0 < \delta < \frac{R}{2}$ s.t

$$d(f(z), f(z')) < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$ and $z, z' \in K_1$ with $|z - z'| < \delta$.

Let $D = \{z_n : z_n \in K_1\}$.

If $z \in K$ then $\exists\ z_n$ with $|z - z_n| < \delta$. But $\delta < \frac{R}{2}$
$\Rightarrow d(z_n, K) < \frac{R}{2}$. Thus $z_n \in K_1$.

As z is arbitrary in K, we have $\{B(w; \delta) : w \in D\}$
is an open cover of K. As K is compact, $\exists$ a
finite cover of K, i.e., $\exists\ w_1, \cdots, w_n \in D$ s.t

$$K \subset \bigcup_{i=1}^{n} B(w_i; \delta).$$

Since $\lim_{k \to \infty} f_k(w_i)$ exists for $1 \le i \le n$, there $\exists\ J > 0$
s.t $j, k \geq J \Rightarrow d(f_k(w_i), f_j(w_i)) < \frac{\varepsilon}{3}$ for all
$1 \le i \le n$.

Let z be an arbitrary pt in K. Then $\exists$ some
$1 \le i \le n$ s.t $z \in B(w_i, \delta)$. Then
$k, j \geq J \Rightarrow$

$$d(f_k(z), f_j(z)) \le d(f_k(z), f_k(w_j)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z))$$
$$< \varepsilon.$$

This proves the lemma.

By the lemma, $\{f_k\}$ has a pointwise limit f s.t
$f_k \to f$ uniformly on K. By proposition 1.10,

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$f_k \to f$ in $C(G, \mathbb{C})$.