Defn: Let G be an open subset of $\mathbb{C}$. Denote by $H(G)$ the set of all analytic functions on G. we will regard $H(G)$ as a subset of $C(G,\mathbb{C})$.

Q: Is $H(G)$ closed in $(C(G,\mathbb{C}),\rho)$?

The answer is yes.

Thm 2.1. If $\{f_n\}$ is a sequence in $H(G)$ and $f\in C(G,\mathbb{C})$ satisfies $f_n\to f$ in $C(G,\mathbb{C})$. then f is analytic and $f_n^{(k)}\to f^{(k)}$ in $C(G,\mathbb{C})$ for each $k\geq 1$.

Pf: Step 1: To show f is analytic, we apply Morera's Thm.

Let T be a triangle in G. Since f_n is analytic,

$$\int_T f_n = 0$$

Since T is compact. $\{f_n\}$ converges to f uniformly on T.

$$\Rightarrow \quad \int_T f = \int_T \lim f_n = \lim \int_T f = 0$$

Thus by Morera's Thm, f is analytic in G.

Step 2: prove $f_n^{(k)}\to f^{(k)}$ in $C(G,\mathbb{C})$.

Fix a disc $D=\overline{B(a;r)}\subset G$. Then $\exists R>r$ s.t

$\overline{B(a;R)}\subset G$

$$\overline{B(a;R)} \subset G$$

Let $\gamma = \{z : |z-a| = r\}$. Then by Cauchy Integra formula,

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\,dw$$

for $z \in D$.

$$\Rightarrow \quad |f_n^{(k)}(z) - f^{(k)}(z)| \le \frac{k!\,M_n R}{(R-r)^{k+1}} \quad \text{for } |z-a| \le r \quad (*).$$

Where $M_n = \sup\{|f_n(w) - f(w)| : |w-a| = R\}$.

Since $f_n \to f$ in $C(G, \mathbb{C})$, $\Rightarrow \lim M_n = 0$.

It follows from $(*)$ that $f_n^{(k)} \to f^{(k)}(z)$ uniformly on $\overline{B(a;r)}$. Note a compact subset K is covered by finitely many discs. we conclude $f_n^{(k)} \to f^{(k)}$ uniformly on K.

Corollary 2.3. $H(G)$ is complete metric space.

Corollary 2.4: If $f_n : G \to \mathbb{C}$ analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compacts to $h(z)$, then

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Remark: The Thm is not true for real analytic functions.

E.g. $h_n(x) = \frac{x^n}{n}$. $0 \le x \le 1$

Then $h_n \to 0$ uniformly

But $\{h_n'\}$ does not converge uniformly on $[0,1]$

Thm 2.5 Hurwitz's Thm

Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f. If $f \not\equiv 0$, $\overline{B(a;R)} \subset G$, and $f(z) \neq 0$ for $|z-a| = R$, then $\exists\ N \in \mathbb{Z}^+$ s.t for $n \ge N$, f and f_n have the same number of zeros in $B(a;R)$.

Pf: Since $f(z) \neq 0$ for $|z-a| = R$,

$$\delta \triangleq \inf\{|f(z)| : |z-a| = R\} > 0.$$

But $f_n \to f$ uniformly on $\{z : |z-a| = R\} \Rightarrow$ There $\exists N$ s.t if $n \ge N$ and $|z-a| = R$, then

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| \le |f(z)| + |f_n(z)|$$

By Rouché's Thm, f_n and f have the same number of zeros in $B(a;R)$.

Corollary 2.6. If G is a region and $\{f_n\} \subset H(G)$

Corollary 2.6. If G is a region and $\{f_n\} \subset H(G)$ converges to f in $H(G)$ and each f_n never vanishes on G $\Rightarrow$ either $f \equiv 0$ or f never vanishes.

Defn 2.7. A set $\mathcal{F} \subset H(G)$ is *locally bdd* if for each pt $a \in G$. there $\exists$ $M>0$, $r>0$ s.t for all $f \in \mathcal{F}$ and $|z-a|<r$, we have $|f(z)| \le M$.

Remark: A set $\mathcal{F}$ in $H(G)$ is locally bdd $\Leftrightarrow$ for every compact $K \subset G$, $\exists$ $M>0$. s.t
$|f(z)| \le M$ for all $f \in \mathcal{F}$ and $z \in K$.

Thm 2.9. (Montel's Thm) A family $\mathcal{F} \in H(G)$ $\Longleftrightarrow$ $\mathcal{F}$ is locally bdd.

pf "$\Rightarrow$" Read by yourself. P153

"$\Leftarrow$" Suppose $\mathcal{F}$ is locally bdd. By Arzela-Ascoli Thm, it suffices to prove $\mathcal{F}$ is equicontinuous at every pt in G. Fix a pt $a \in G$ and $\varepsilon > 0$. Since $\mathcal{F}$ is locally bdd, $\exists$ $r>0$ and $M>0$ s.t

$$\overline{B(a;r)} \subset G \text{ and } |f(z)| \le M \text{ for all } z \in \overline{B(a;r)}, f \in \mathcal{F}.$$

Now let $|z-a| < \frac{r}{2}$ and $f \in \mathcal{F}$, and $\gamma = \{w \mid |w-a| = r\}$.

Now let $|z-a|<\frac{r}{2}$ and $f\in\mathcal{F}$, and $\gamma=\{w \mid |w-a|=r\}$.

$$\Rightarrow \quad |f(a)-f(z)| = \frac{1}{2\pi}\left|\int_\gamma \left(\frac{f(w)}{w-a} - \frac{f(w)}{w-z}\right) dw\right|$$

$$= \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}\, dw\right|$$

$$\leq \frac{2M}{r}|a-z|$$

There $\exists\, \delta<\frac{r}{2}$ s.t

when $|a-z|<\delta \Rightarrow |f(a)-f(z)|<\varepsilon$ for all $f\in\mathcal{F}$.

Corollary 2.10. A set $\mathcal{F}\in H(G)$ is compact $\Leftrightarrow$ $\mathcal{F}$ is closed and locally bdd.

7.3 spaces of meromorphic functions

Recall the metric on $\mathbb{C}_\infty$:

(a) If $z_1, z_2\in\mathbb{C} \Rightarrow d(z_1,z_2) = \dfrac{2|z_1-z_2|}{[(1+|z_1|^2)(1+|z_2|^2)]^{\frac{1}{2}}}$

(b) If $z_1=z\in\infty$, $z_2=\infty$

$$d(z,\infty)=\frac{2}{(1+|z|^2)^{\frac{1}{2}}}.$$

Note: Let $\{z_n\}\subset\mathbb{C}$, $z\in\mathbb{C}$

$$d(z_n,z)\to 0 \Leftrightarrow |z_n-z|\to 0$$

__Defn__: Let $M(G)$ = the set of all meromorphic functions

Def^n: Let $M(G)$ = the set of all meromorphic functions on G.

Recall. every meromorphic function f on G can be identified with a new map $\hat{f}: G \to \mathbb{C}$:

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \text{ is not a pole} \\ \infty & \text{if } z \text{ is a pole.} \end{cases}$$

Hence $M(z)$ can be regarded as a subset of $C(G, \mathbb{C}_\infty)$.

Metric on $C(G, \mathbb{C}_\infty)$.

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}$$

Here $\{K_n\}_{n=1}^{\infty}$ is an exhaustion sequence of G and

$$\rho_n(f, g) = \sup\{d(f(z), g(z)) : z \in K_n\}.$$

We say $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ if $\rho(f_n, f) \to 0$.

Thm 3.4 Let $\{f_n\}$ be a sequence in $M(G)$ and $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Then either f is meromorphic or $f \equiv \infty$. If each f_n is analytic then either f is analytic or $f \equiv \infty$

Corollary 3.5: $M(G) \cup \{\infty\}$ is a complete metric space

Corollary 3.5: $M(G) \cup \{\infty\}$ is a complete metric space

Corollary 3.6: $H(G) \cup \{\infty\}$ is closed in $C(G, \mathbb{C}_\infty)$.

Ex Read 7.3.