7.4 The Riemann Mapping Thm

Defn 4.1: A region G_1 is conformally equivalent to G_2 if there $\exists$ an analytic function $f: G_1 \to \mathbb{C}$ such that f is one-one and $f(G_1) = G_2$.

Remark: Conformal equivalence is an equivalence relation.

E.g Let $G_1 = \mathbb{C} - \{z \in \mathbb{R} : z \le 0\}$

$G_2 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

Let $f(z) = \sqrt{z}$ the principle branch of the square root.

Thm 4.2 (Riemann Mapping Thm) Let G be a simply connected region with $G \neq \mathbb{C}$. Let $a \in G$. Then $\exists$ a unique analytic function $f: G \to \mathbb{C}$ having the properties:

(a) $f(a) = 0$ and $f'(a) > 0$

(b) f is 1-1

(c) $f(G) = \{z : |z| < 1\}$.

pf: The uniqueness part is easy: If f, g both satisfy (a), (b), (c). Then $f \circ g^{-1} : D \to D$ is analytic, 1-1, onto.

$f \circ g^{-1} : D \to D$ is analytic, ...

Also $f \circ g^{-1}(0) = 0$.

By Schwarz's lemma, $f \circ g^{-1} = cz$ where $|c| = 1$.

$\Rightarrow \quad f = c\,g(z)$.

Differentiate at $z = a$; $\Rightarrow$

$$f'(a) = c\,g'(a).$$

Note $f'(a) > 0$, $g'(a) > 0 \Rightarrow c = 1$.

Thus $f = g$.

Step 1: We first prove $\exists\, \varphi \in H(G)$, 1-1 on G. s.t

$\varphi(G)$ is bdd.

Note as $G \neq \mathbb{C}$, $\exists\, b \in \mathbb{C}$ s.t $b \notin G$. Thus $h \triangleq z - b$ never vanishes in G. Recall on P94, Corollary 6.17.

If G is simply connected, $h \in H(G)$ and $h(z) \neq 0$, $\forall z \in G$, then $\exists\, \hat{h} \in H(G)$ s.t $h = e^{\hat{h}}$. Moreover, $\sqrt{w}$ can be defined by $e^{\frac{1}{2}\log w}$. Hence

$\sqrt{z-b}$ has a well-defined branch in G

Call it $g(z)$. Note g is 1-1.

Moreover, write $g(G) = E$. Then $0 \notin E$.

Claim: If $w \in E$, $-w \notin E$.

Pf: Suppose $w, -w \in E$, then $\exists\, z_1, z_2$, s.t

$$w = \sqrt{z_1 - b}, \quad -w = \sqrt{z_2 - b}$$

$$\Rightarrow -\sqrt{z_1 - b} = \sqrt{z_2 - b} \Rightarrow z_1 - b = z_2 - b$$

$$\Rightarrow z_1 = z_2 \Rightarrow w = -w \Rightarrow w = 0$$

$\Rightarrow z_1 = z_2 \Rightarrow w = -w \Rightarrow w = 0$

This is a contradiction.

By open mapping thm. E is a region in $\mathbb{C}$.

As $a \in G \Rightarrow g(a)$ is an interior pt of E

$\Rightarrow \exists\, \delta > 0$. s.t $B(g(a);\delta) \subset E$.

Thus $B(-g(a);\delta) \subset \mathbb{C} - E \Rightarrow \operatorname{dist}(-g(a), E) \geq \delta$

Thus when $z \in G$, we have $|g(z) + g(a)| \geq \delta$.

$$\text{Let } \varphi(z) = \frac{1}{g(z) + g(a)}$$

Thus if $z \in G$ $\quad |\varphi(z)| \leq \frac{1}{\delta}$ Thus $\varphi(G)$ is bdd.

Note: φ is 1-1. as $g(z)$ is 1-1.

Step 2: Assume G is bdd.

To prove the existence,

Idea: Consider the family $\mathcal{F}$ of all analytic functions having properties (a). (b). and maps G to $\mathbb{D}$.

Then choose a special element f in $\mathcal{F}$ satisfies (c).

Let
$$\mathcal{F} = \{ f : f \in H(G),\ 1\text{-}1,\ f(a) = 0,\ f'(a) > 0,\ f(G) \subset \mathbb{D} \}.$$

Claim: $\mathcal{F} \neq \phi$.

Pf: Assume $G \subset B(a;R)$ for some $R >> 0$.

Let $h(z) = \frac{z-a}{R}$. Then $h \in \mathcal{F}$.

... By Cauchy's Estimate

Now fix $r>0$. s.t $B(a;r) \subset G$. By Cauchy's Estimate, we obtain for $\forall f \in \mathcal{F}$,

$$f'(a) = |f'(a)| \leq \frac{1}{r} \sup\{|f(z)| : z \in B(a;r)\} \leq \frac{1}{r}$$

Thus $\{f'(a) : f \in \mathcal{F}\}$ has an upper bound. $\Longrightarrow$

$$M = \sup\{f'(a) : f \in \mathcal{F}\} < \infty. \text{ Note } M>0.$$

Next we will prove $f_* \in \mathcal{F}$ such that $f_*'(a) = M$. Indeed, $\exists \{f_n\} \in \mathcal{F}$ such that $\lim_{n\to\infty} f_n'(a) = M$. But $|f(z)| \leq 1$ on G for all $f \in \mathcal{F}$. By Montel's thm. we have

$\{f_n\}$ has a convergent subsequence in $H(G)$. we still call it $\{f_n\}$. Write the limit as f_*:

$$f_n \to f_* \text{ in } H(G).$$

We also have $f_n' \to f_*'$ in $H(G)$. In particular,

$$f_*'(a) = \lim_{n\to\infty} f_n'(a) = M > 0$$

As each f_n is 1-1, $f \not\equiv$ constant, by what we proved last time,

f_* is also 1-1.

Moreover, $f_*(a) = \lim_{n\to\infty} f_n(a) = 0$.

<u>Claim</u>: $|f_*(z)| < 1$ for $z \in G$.

Pf: As $|f_n(z)| < 1$ for any $z \in G$, $n \geq 1$

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Let $n \to \infty$. $\Rightarrow$ $|f(z)| \leq 1$ for $z \in G$.

Note f is not constant. By M.M.T

$|f(z)| < 1$ for $z \in G$.

It remains to prove $f(G) = \mathbb{D}$.

Step 3: It remains to show that $f(G) = D$. Suppose $w \in D$ s.t $w \notin f(G)$. Then the function

$$\frac{f(z) - w}{1 - \bar{w} f(z)} = \varphi_w \circ f$$

is analytic in G and never vanishes. Then $\exists$ an analytic function $h: G \to \mathbb{C}$ such that

$$h^2 = \frac{f - w}{1 - \bar{w} f} \qquad (*)$$

Note h^2 maps G to $\mathbb{D}$, is 1-1 $\Rightarrow$ h maps G to $\mathbb{D}$, is 1-1.

Define $g: G \to \mathbb{C}$ by

$$g(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)}$$

Then $g(G) \subset D$. $g(a) = 0$, and is 1-1

Also
$$g'(a) = \frac{|h'(a)|}{h'(a)} \frac{h'(a)[1 - |h(a)|^2]}{[1 - |h(a)|^2]^2}$$
$$= \frac{|h'(a)|}{1 - |h(a)|^2}$$

But $f(a) = 0 \Rightarrow |h(a)|^2 = |w|$.

But $f(a)=0 \Rightarrow |h(a)|^2 = |w|$.

Differentiate $(*) \Rightarrow$

$$2h(a)\,h'(a) = f'(a)\,(1-|w|^2)$$

$$\Rightarrow \qquad |h'(a)| = \frac{f'(a)\,(1-|w|^2)}{2\sqrt{|w|}}$$

$$\Rightarrow \qquad g'(a) = \frac{f'(a)\,(1-|w|^2)}{2\sqrt{|w|}} \, \frac{1}{1-|w|}$$

$$= f'(a)\,\frac{1+|w|}{2\sqrt{|w|}}$$

$$> f'(a).$$

Note $g \in \mathcal{F}$ and this contradicts with the choice of f.

Hence it must hold that $f(G) = \mathbb{D}$.