The Weierstrass Factorization Thm.

Q: Given a sequence $\{a_k\}$ in G which has no limit pt in G and a sequence of intergers $\{m_k\}$, is there a function $f \in H(G)$ s.t the only zeros of f are the pts a_k. Moreover,

the multiplicity of zero at $a_k = m_k$.

The answer is yes by a thm of Weierstrass.

Corollary 5.20. If f is a meromophic function on an open set G, then $\exists\, g, h \in H(G)$ s.t $f = \frac{g}{h}$.

Pf: Let $\{a_j\}$ be the poles of f and m_j be the order of the pole at a_j. According to the thm of Weierstrass, $\exists\, h \in H(G)$ with a zero of multiplicity m_j at each $z = a_j$ and with no other zeros. Thus hf has removable singularities at each pt a_j. It follows that $g = hf$ is analytic in G.

We will prove the thm of Weierstrass. in this section

Defn. If $\{z_n\}$ is a sequence of complex numbers and if $z = \lim \prod_{n=1}^{N} z_n$ exists, then z is called the

If $z = \lim \prod_{n=1}^{N} z_n$ exists, then z is called the infinite product of the numbers z_n. It is denoted by

$$z = \prod_{n=1}^{\infty} z_n .$$

Remark:

① If some $z_n = 0$. Then $\prod_{n=1}^{N} z_n = 0$ for $N >> 1$.

② If every $z_n \neq 0$, suppose $z = \prod_{n=1}^{\infty} z_n$ exists and is not zero. Let $p_n = \prod_{k=1}^{n} z_k$ for $n \geq 1$. Then

$$p_n \neq 0 \quad \text{and} \quad \frac{p_n}{p_{n-1}} = z_n .$$

Since $z \neq 0$ and $p_n \to z$ we have that $\lim z_n = 1$.

In this case. $\operatorname{Re} z_n > 0$ for large n.

We will be interested in ②. Hence we will always assume $\operatorname{Re} z_n > 0$ for all n.

Let log be the principal branch of the logrithm.

Then $\log z_n$ is well-defined.

Consider $\sum_{n=1}^{\infty} \log z_n$. Suppose $\sum_{n=1}^{\infty} \log z_n$ exists.

Write $S_N = \sum_{n=1}^{N} \log z_n$ and $S_N \to s$. Then

$$\exp S_N \to \exp s$$

But $\exp S_N = \prod_{n=1}^{N} S_n \Rightarrow \prod_{n=1}^{\infty} z_n$ is convergent to z

s.

But $\exp S_N = \prod_{n=1} z_n \Rightarrow \prod_{n=1} z_n$ is convergent ...

Moreover, $z = e^s \neq 0$.

Indeed, we have

Proposition 5.2: Let $\mathrm{Re}\, z_n > 0$ for all $n \geq 1$. Then $\prod_{n=1}^{\infty} z_n$ converges to a nonzero number iff the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Pf: Read P165

Consider the power series expansion of $\log(1+z)$ about $z=0$:

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \cdots$$

which has radius of convergence 1. If $|z|<1 \Rightarrow$

$$\left| 1 - \frac{\log(1+z)}{z} \right| = \left| \frac{1}{2} z - \frac{1}{3} z^2 + \cdots \right|$$
$$\leq \frac{1}{2} \left(|z| + |z|^2 + \cdots \right)$$
$$= \frac{1}{2} \frac{|z|}{1-|z|}.$$

If we further require $|z| < \frac{1}{2} \Rightarrow$.

$$\left| 1 - \frac{\log(1+z)}{z} \right| \leq \frac{1}{2}$$

This implies for $|z| < \frac{1}{2}$

$$\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z| \qquad (1)$$

$$\frac{1}{2}|z| \le |\log(1+z)| \le 2|z| \qquad (1)$$

proposition 5.4: Let $\operatorname{Re} z_n > -1$. Then the series $\sum \log(1+z_n)$ converges absolutely $\Longleftrightarrow$ the series $\sum z_n$ converges absolutely

Pf: If $\sum |z_n|$ converges then $z_n \to 0$. Thus $|z_n| < \frac{1}{2}$ for large n. By (1) $\Rightarrow \sum |\log(1+z_n)|$ converges.

Conversely, assume $\sum |\log(1+z_n)|$ converges. Then it follows that $\log(1+z_n) \to 0 \Rightarrow z_n \to 0$

Thus $|z_n| < \frac{1}{2}$ for $n >> 1$. By (1) again, $\sum |z_n|$ converges.

Next we define $\prod z_n$ converges absolutely We cannot define it in the way that $\prod |z_n|$ converges.

E.g $z_n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$ then $\prod |z_n|$ converges but $\prod z_n$ does not converge.

Defn. If $\operatorname{Re} z_n > 0$ for all n, then the infinite product $\prod z_n$ is said to converge absolutely if $\sum \log z_n$ converges absolutely

Corollary 5.6. If $\operatorname{Re} z_n > 0$, then the product $\prod z_n$ converges absolutely $\Longleftrightarrow \sum (z_n - 1)$ converges

converges absolutely $\Longleftrightarrow \sum (z_n - 1)$ converges absolutely.

Lemma 5.7: Let X be a set and let $f, f_1, \ldots$ be functions from X into $\mathbb{C}$ s.t $f_n \to f$ uniformly for $x \in X$. If $\exists$ a constant a s.t $\operatorname{Re} f(x) \le a$ for all $x \in X$, then

$$\exp f_n(x) \to \exp f(x) \text{ uniformly for } x \in X.$$

Pf: If $\varepsilon > 0$ is given then $\exists \delta > 0$ s.t

$$|e^z - 1| < \varepsilon e^{-a} \text{ whenever } |z| < \delta.$$

Now choose n_0 s.t $|f_n(x) - f(x)| < \delta$ for all $x \in X$. whenever $n \ge n_0$. Thus

$$\begin{aligned} \varepsilon e^{-a} &> \left| \exp[f_n(x) - f(x)] - 1 \right| \\ &= \left| \frac{\exp f_n(x)}{\exp f(x)} - 1 \right| \end{aligned}$$

It follows that for any x in X and for $n \ge n_0$.

$$|\exp f_n(x) - \exp f(x)| < \varepsilon e^{-a} |\exp f(x)| \le \varepsilon.$$

Lemma 5.8: Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous function from X in $\mathbb{C}$ s.t $\sum g_n(x)$ converges absolutely and uniformly for $x \in X$. Then the product

$$f(x) = \prod_{n=1}^{\infty} (1 + g_n(x))$$

$$f(x) = \prod_{n=1} (1+g_n(x))$$

converges absolutely and uniformly for $x \in X$.

Also $\exists$ an integer n_0 s.t

$$f(x) = 0 \iff g_n(x) = -1$$

Pf: Since $\sum g_n(x)$ converges uniformly for $x \in X$

$\exists\, n_0$ s.t $|g_n(x)| < \frac{1}{2}$ for all $x \in X$ and $n > n_0$

$\Rightarrow \quad \mathrm{Re}[1+g_n(x)] > 0$

By (1), $\quad |\log(1+g_n(x))| \le \frac{3}{2}|g_n(x)|$ for $n > n_0$

Thus
$$h(x) = \sum_{n=n_0+1}^{\infty} \log(1+g_n(x))$$

converges uniformly for $x \in X$

Since h is continuous and X is compact $\Rightarrow$

h must be bdd.

In particular, $\exists\, a$ s.t $\mathrm{Re}\, h(x) < a$ for $x \in X$.

By lemma 5.7,
$$\exp h(x) = \prod_{n=n_0+1}^{\infty} (1+g_n(x))$$

converges uniformly for $x \in X$.

Finally, $\quad f(x) = [1+g_1(x)] \cdots [1+g_{n_0}(x)] \exp h(x)$

and $\exp h(x) \neq 0$ for $x \in X$

Hence if $f(x) = 0$, there $\exists\, g_n(x) = -1$. for $1 \le n \le n_0$

Thm 5.9. Let G be a region in $\mathbb{C}$ and let $\{f_n\} \subset H(G)$ s.t

Thm 5.9: Let G be a region in $\mathbb{C}$ and let $\{f_n\} \subset H(G)$ s.t no f_n is identically zero. If $\sum [f_n(z)-1]$ converges absolutely and uniformly on compact subsets of G then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of the functions f_n and the multiplicity at a of f is the sum of the multiplicities at a of f_n.

Back to the question at the beginning, the idea is:

Let $\{a_n\}$ is a sequence in a region G with no limit pt in G (but some pt may be repeated in the sequence a finite number of times). If we can functions $g_n(z)$ which are analytic on G, have no zeros in G such that

$$\sum |(z-a_n)g_n(z)-1| \text{ converges uniformly on}$$

compact subsets of G

Then by Thm 5.9, $\prod (z-a_n)g_n$ is analytic and has its zeros only at $z=a_n$

Defn: An elementary factor is one of the following function $E_p(x)$ for $p=0,1,\cdots$;

$$E_0(z) = 1-z,$$

$$E_p(z) = (1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right); \; p \geq 1.$$

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right); \ p \geq 1.$$

Lemma 5.11. If $|z| \leq 1$ and $p \geq 0$ then $|1 - E_p(z)| \leq |z|^{p+1}$.

Pf: Read P168

Thm 5.12. Let $\{a_n\}$ be a sequence in $\mathbb{C}$ s.t $\lim |a_n| = \infty$ and $a_n \neq 0$ for all $n \geq 1$. (some pt may be repeated finitely many times, but not infinitely many times)

If $\{p_n\}$ is any sequence of integers s.t

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \text{ for all } r > 0, \quad (2)$$

then

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \text{ converges in } H(\mathbb{C}).$$

The function f is an entire function with zeros only at the pts a_n. If the pt z_0 occurs in the sequence $\{a_n\}$ exactly m times then f has a zero at $z = z_0$ of multiplicity m.

Remark: Note we can take $p_n = n-1$ in $(*)$.

Pf: Suppose there are integers p_n s.t (2) is satisfied.

Then, by Lemma 5.11,

$$\left|1 - E_{p_n}\left(\frac{z}{a_n}\right)\right| \leq \left|\frac{z}{a_n}\right|^{p_n+1} \leq \left(\frac{r}{|a_n|}\right)^{p_n+1}$$

where $|z| \leq r$. For a fixed $r > 0$, $\exists N$ s.t $|a_n| \geq r$ for $n \geq N$ (as $\lim |a_n| = \infty$). Thus for each $r > 0$

$n \geq N$ (as $\lim |a_n| = \infty$). Thus for each $r > 0$

$\sum |1 - E_{p_n}(\frac{z}{a_n})|$ is dominated by the series in $(*)$ on $\overline{B(a;r)}$. $\Rightarrow$

$\sum [1 - E_{p_n}(\frac{z}{a_n})]$ converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

By Thm 5.9, $\prod_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$ converges in $H(\mathbb{C})$

Thm 5.14. (The Weierstrass Factorization Thm) Let f be an entire function and let $\{a_n\}$ be the non-zero zeros of f repeated to multiplicity; suppose f has a zero at $z=0$ of order $m \geq 0$. (If $m=0$, it means $f(0) \neq 0$). Then $\exists$ an entire function g and a sequence of integers $\{p_n\}$ s.t

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Pf: By Thm 5.12, $\exists$ integers $\{p_n\}$ s.t

$$h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

has the same zeros as f with the same multiplicities. It follows that $\frac{f(z)}{h(z)}$ has removable singularities at $z=0, a_1, a_2, \cdots$. Thus $\frac{f}{h}$ is an entire function and furthermore, has no zeros. Since $\mathbb{C}$ is simply connected, $\exists$ an entire function g s.t

$$\frac{f(z)}{h(z)} = e^{g(z)}.$$

This proves the thm.

Thm 5.15. Let G be a region and let $\{a_j\}$ be a sequence

Thm 5.15'. Let G be a region and let $\{a_j\}$ be a sequence of distinct pts in G with no limit pt in G; and let $\{m_j\}$ be a sequence of integers. Then $\exists$ an analytic function f defined on G whose only zeros at the pts a_j; furthermore, a_j is a zero of f of multiplicity m_j.

Pf: Read P170.

7.6 Factorization of the sine function

Notation: $\sum'$ or $\prod'$ means the sum or the product is taken over all indicated n except $n=0$.

E.g
$$\sum_{n=-\infty}^{\infty}{}' a_n = \sum_{n=1}^{\infty} a_{-n} + \sum_{n=1}^{\infty} a_n.$$

We use the Weierstrass Factorization Thm to study the function $\sin\pi z = \frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right)$

Note the zeros of $\sin\pi z$ are precisely the integers. Moreover, each zero is simple.

Since $\sum_{n=-\infty}^{+\infty}{}' \left(\frac{r}{n}\right)^2 < \infty$ for all $r>0$.

one can choose $p_n = 1$ in (2) in the Weierstrass Factorization Thm. Thus
$$\sin\pi z = [\exp g(z)]\, z \prod_{n=-\infty}^{+\infty}{}' \left(1-\frac{z}{n}\right)e^{\frac{z}{n}}$$

or
$$\sin\pi z = [\exp g(z)]\, z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

for some entire function $g(z)$. If $f(z) = \sin\pi z$ then,
$$\pi\cot\pi = \frac{f'(z)}{f(z)} = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

The convergence is uniform over compact subsets of $\mathbb{C} - \{z\in\mathbb{C} : z \text{ integer}\}$.

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Claim: $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ for z not an integer.

$\Rightarrow$ g is a constant: $g \equiv a$.

$$\Rightarrow \quad \frac{\sin \pi z}{\pi z} = \frac{e^a}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Let $z \to 0 \Rightarrow e^a = \pi$. Thus

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

and the convergence is uniform over compact subsets of $\mathbb{C}$.