1. Runge's Thm

Recall: If D is a disk and $f \in H(D)$, then we know f can be written as a power series that converges uniformly on every compact subset of D, i.e converges in $H(D)$. $\Rightarrow f$ is the limit of a sequence of polynomials in $H(D)$. we can thus natural ask:

Q: Let $f \in H(G)$ for a region G. Is f the limit of a sequence of polynomials in $H(G)$.

A: No in general.

E.g $G = B(0;2) - \{0\}$, $f = \frac{1}{z}$

Suppose true, i.e., $\exists \{p_n\}$ s.t $p_n \to f$ in $H(G)$.

Take $\gamma = \{|z|=1\}$. Then

$$\int_\gamma p_n = 0 \text{ for all } n. \quad \text{But } p_n \to f \text{ in } H(G)$$

$$\Rightarrow \int_\gamma f = \int_\gamma \lim p_n = \lim \int_\gamma p_n = 0$$

This is a contradiction.

Indeed, inspaired by Laurent series expansion, it makes more sense if we use the limit of rational functions instead of polynomial.

of polynomial.

Thm. Let K be a compact subset of $\mathbb{C}$ and let E be a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$. If f is analytic in an open set containing K and $\varepsilon > 0$ then $\exists$ a rational function $R(z)$ whose poles lie in E and s.t

$$|f(z) - R(z)| < \varepsilon \quad \text{for all } z \in K.$$

Corollary 1.14: Let $G \subset \mathbb{C}$ open and let E be a subset of $\mathbb{C}_\infty - G$ s.t E meets every component of $\mathbb{C}_\infty - G$. Let $R(G,E)$ = the set of rational functions with poles in E. $\subset H(G)$. If $f \in H(G)$ then $\exists$ a sequence $\{R_n\}$ in $R(G,E)$ s.t $f = \lim R_n$. That is, $R(G,E)$ is dense in $H(G)$.

Pf: Let K be a compact subset of G and $\varepsilon > 0$; it must be shown that $\exists\, R \in R(G,E)$ s.t $|f(z) - R(z)| < \varepsilon$ for $z \in K$.

claim: there is compact $\hat{K} \subset G$ s.t $K \subset \hat{K}$ and each component of $\mathbb{C}_\infty - \hat{K}$ contains a component of $\mathbb{C}_\infty - G$.

Pf: Exercise or read Chapter 7. proposition 1.2.

By the claim, E meet each component of $\mathbb{C}_\infty - \hat{K}$. The corollary now follows from Runge's Thm.

The corollary now follows from Runge's Thm.

Corollary 1.15: If G is an open subset of $\mathbb{C}$ s.t $\mathbb{C}_\infty - G$ is connected then for each analytic $f \in H(G)$, $\exists$ a sequence of polynomials $\{p_n\}$ s.t $f = \lim p_n$ in $H(G)$.

Pf: Let $E = \{\infty\}$ and apply Corollary 1.14.

Remark: The condition that E meets every component of $\mathbb{C}_\infty - G$ cannot be relaxed.

E.g Let $G = \mathbb{C} - \{0\}$, thus $\mathbb{C}_\infty - G = \{0, \infty\}$

Let $E = \{\infty\}$ $\quad f = \frac{1}{z} \in H(G)$

Suppose the corollary is still true, i.e

$\exists \{p_n\}$ polynomial s.t $p_n \to \frac{1}{z}$ in $H(G)$

we know this is impossible.

Idea of proof of Runge's Thm:

we will write

$B(E) = \{$all function f in $C(K, \mathbb{C})$ s.t $\exists$ a sequence $\{R_n\}$ of rational functions with poles in E s.t $f = u\text{-}\lim f_n$ on $K\}$

Then Runge's Thm asserts if f is analytic in a nbhd of K, then $f|_K \in B(E)$

We observe:

Lemma 1.8: $B(E)$ is a closed subalgebra of $C(K, \mathbb{C})$ that contains every rational function with poles in E.

Pf: ① closedness: That is, if $f_n \to f$ in $C(K, \mathbb{C})$ and $f_n \in B(E)$, then $f \in B(E)$. Indeed, as $f_n \in B(E)$, $\exists R_n$, rational with poles in E, s.t

$$|f_n - R_n| < \frac{1}{n} \quad \text{on } K$$

Then $\quad f = u\text{-}\lim R_n$.

② subalgebra: if $f, g \in B(E)$, then αf, $f+g$, fg are in $B(E)$ for $\alpha \in \mathbb{C}$. We will leave this as an exercise.

Lemma 1.9: Let V and U be open subsets of $\mathbb{C}$ with $V \subset U$ and $\partial V \cap U = \phi$. If H is a component of U and $H \cap V \neq \phi$ then $H \subset V$.

pf. We will only prove for the case U is connected. Then by assumption $U \cap V \neq \phi \Rightarrow V \neq \phi$. If we can prove V is open and closed in U, then $V = U$.

openness is clear.

closedness follows from the assumption $\partial V \cap U = \phi$.

We prove Runge's Thm.

Step 1. If f is analytic in a nbhd G of K, then for

Step 1: If f is analytic in a nbhd G of K, then for each $\varepsilon > 0$, $\exists$ $R(z)$ rational with poles in $\mathbb{C}-K$ s.t $|f(z) - R(z)| < \varepsilon$ on K.

Step 2. If $R(z)$ is rational with poles in $\mathbb{C}-K$, then for each $\varepsilon>0$, $\exists$ $\hat{R}(z)$ rational with poles in E s.t

$$|R(z) - \hat{R}(z)| < \varepsilon.$$

That is. $R(z) \in B(E)$.

Claim: $R(z)$ is in the algebra generated by polynomials together with $\{(z-a)^{-1} : a \in \mathbb{C}-K\}$

Pf: Exercise.

Then to show $R(z) \in B(E)$, by Lemma 1.8 and the claim, it suffices to show:

Lemma 1.10: If $a \in \mathbb{C}-K$, then $(z-a)^{-1} \in B(E)$.

Pf: Case 1. If $\infty \notin E$.

Let $U \triangleq \mathbb{C}-K$, and $V \triangleq \{a \in \mathbb{C} : (z-a)^{-1} \in B(E)\}$.

Note $E \subset U$. Also $V \subset U$.

① V is open

Claim: If $a \in V$ and $|b-a| < d(a;K)$ then $b \in V$

Pf: Indeed, we need to prove $\frac{1}{z-b} \in B(E)$.

Note since $|b-a| < d(a,K)$, $\exists\ 0<r<1$, s.t

$$|b-a| < r|z-a| \quad \text{for } \forall z \in K.$$

$$\Rightarrow \quad \frac{1}{z-b} = (z-a)^{-1}\left[1-\frac{b-a}{z-a}\right]^{-1} \qquad (*)$$

But $\left[1-\frac{b-a}{z-a}\right]^{-1} = \sum_{k=0}^{\infty}\left(\frac{b-a}{z-a}\right)^k$ converges uniformly on K.

Let $Q_n(z) = \sum_{k=0}^{n}\left(\frac{b-a}{z-a}\right)^k$. Then $Q_n(z) \in B(E)$.

As $B(E)$ is closed in $C(K,\mathbb{C}) \Rightarrow$

$$\lim_{n\to\infty} Q_n(z) = \left[1-\frac{b-a}{z-a}\right]^{-1} \in B(E)$$

As $(z-a)^{-1} \in B(E)$, by $(*)$ $\Rightarrow (z-b)^{-1} \in B(E)$.

This proves the claim and the openness of V follows.

② $\partial V \cap U = \phi$.

Let $b \in \partial V$. Then $\exists \{a_n\} \subset V$ s.t $\lim a_n = b$.

Note V is open $\Rightarrow b \notin V$. By the claim above,

$$|b-a_n| \geq d(a_n,K)$$

Let $n\to\infty$

$$\Rightarrow \quad d(b,K) = 0$$

since K is compact $\Rightarrow b \in K \Rightarrow b \notin U$

$\Rightarrow \partial V \cap U = \phi$.