2. Simply connectedness

Defn. Let X and Ω be metric spaces. A homeomorphism between X and Ω is a continuous map $f: X \to \Omega$ which is one-to-one, onto and s.t $f^{-1}: \Omega \to X$ is also continuous.

If such $f: X \to \Omega$ exists, we say X and Ω are homeomorphic.

Ex Verify $f: \mathbb{C} \to \mathbb{D} = \{|z|<1\}$ given by $f(z) = \frac{z}{1+|z|}$ is a homeomorphism, thus $\mathbb{C}$ and $\mathbb{D}$ are homeomorphic.

Ex prove all annuli are homeomorphic to the punctured disk.

Thm. Let G be an open connected subset of $\mathbb{C}$. Then TFAE:

(a) G is simply connected;

(b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every $a \in \mathbb{C} - G$

(c) $\mathbb{C}_\infty - G$ is connected

(d) For any $f \in H(G)$, $\exists$ a sequence of polynomials that converges to f in $H(G)$.

(e) For any $f \in H(G)$ and any closed rectifiable curve γ in G, $\int_\gamma f = 0$

(f) Every function $f \in H(G)$ has a primitive.

(g) For any $f \in H(G)$ s.t $f(z) \neq 0$ in G, $\exists\, g \in H(G)$

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s.t $f(z) = e^{g(z)}$

(h) For any $f \in H(G)$ s.t $f(z) \neq 0$ in G, $\exists\, g \in H(G)$

s.t $f(z) = [g(z)]^2$.

(i) G is homeomorphic to the unit disk.

(j) If $u: G \to \mathbb{R}$ is harmonic, then $\exists$ a harmonic $v: G \to \mathbb{R}$ s.t $f = u + iv$ is analytic on G.

pf: we will prove (a) $\Rightarrow$ (b) $\Rightarrow \dots \Rightarrow$ (i) $\Rightarrow$ (a) and (h) $\Rightarrow$ (j) $\Rightarrow$ (g).

(a) $\Rightarrow$ (b)

Recall Cauchy's Thm: If $f \in H(G)$ and $\gamma \subset G$ closed rectifible s.t $\gamma \sim 0$, then

$$\int_\gamma f = 0$$

As $n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz \Rightarrow n(\gamma; a) = 0$

$\frac{1}{z-a} \in H(G)$

(b) $\Rightarrow$ (c). Suppose $\mathbb{C}_\infty - G$ is not connected, then $\mathbb{C}_\infty - G = A \cup B$ where A, B are disjoint nonempty, closed subset of $\mathbb{C}_\infty$. As $\infty \in A \cup B$, WLOG, we assume $\infty \in B$. Then A must be a compact subset of $\mathbb{C}$.

But then $G_1 = G \cup A = \mathbb{C}_\infty - B$ is an open set in $\mathbb{C}$ and contains A.

By proposition 1.1, $\exists$ a finite number of polygons $\gamma_1, \cdots, \gamma_m$ in $G_1 - A = G$ s.t for $\forall f \in H(G_1)$

$$f(z) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw$$

for all $z \in A$. In particular, if $f(z) \equiv 1$, then

$$1 = \sum_{k=1}^{m} n(\gamma_k; z) \quad \text{for } z \in A.$$

Thus for any $z \in A$, $\exists$ at least one polygon γ_k in G s.t $n(\gamma_k; z) \neq 0$. This condicts (b).

(c) $\Rightarrow$ (d) See Corollary 1.5. Recall let $E = \{\infty\}$. by Runge's Thm, $\exists$ rational $\{R_N\}$ whose poles in E s.t $R_N \to f$ in $H(G)$.

(d) $\Rightarrow$ (e) Let $\gamma \subset G$ closed rectifible., $f \in H(G)$.

Then $\exists \{P_N\}$ s.t $P_N \to f$ in $H(G)$.

$$\text{As } \int_\gamma P_N = 0 \Rightarrow \int_\gamma f = 0$$

(e) $\Rightarrow$ (f) Fix $a \in G$. Define

$$F(z) = \int_\gamma f(z) \quad \text{where } \gamma \text{ is any}$$

rectifible curve in G from a to z.

Then F is well-defined and $F' = f$ (see the pf of Corollary IV.6.16.

(f) $\Rightarrow$ (g) $\left(g = \log f \Rightarrow g' = \frac{f'}{f} \right)$

As $f \neq 0$ in G, $\Rightarrow \frac{f'}{f} \in H(G)$

By (f), $\exists F \in H(G)$ s.t $F' = \frac{f'}{f}$

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Consider $h = f e^{-F} \Rightarrow$

$$h' = f' e^{-F} - f e^{-F} F'$$
$$= e^{-F}(f' - f F')$$
$$= 0$$

$\Rightarrow h \equiv c \neq 0 \Rightarrow f = c e^{F}$

Let $c = e^{\alpha}$, $\alpha \in \mathbb{C}$

Let $g = \alpha + F \Rightarrow f = e^{g}$.

(g) $\Rightarrow$ (h) By (g), $\exists g \in H(G)$ s.t $f = e^{g(z)}$

Let $h = e^{\frac{g}{2}} \Rightarrow h^2 = f$

(h) $\Rightarrow$ (i) If $G = \mathbb{C}$, then the function $z(1+|z|)^{-1}$ gives a homeomorphism from $\mathbb{C}$ to $\mathbb{D}$.

If $G \neq \mathbb{C}$, by the pf of Riemann mapping thm, $\exists f \in H(G)$ one-to-one, onto $\mathbb{D}$.

(i) $\Rightarrow$ (a) Trivial.

(h) $\Rightarrow$ (j) First recall we have proved (j) for $G = \mathbb{C}$ or $\mathbb{D}$.

Now if $G \neq \mathbb{C}$, then the Riemann Mapping thm, $\Rightarrow$ $\exists h \in H(G)$, s.t h is one-to-one and $h(G) = \mathbb{D}$.

If $u: G \to \mathbb{R}$ harmonic.

claim: $u_1 = u \circ h^{-1}$ is harmonic on $\mathbb{D}$

pf: Exercise.

By chapter III $\exists v_1 : \mathbb{D} \to \mathbb{R}$ s.t $f_1 = u_1 + i v_1 \in H(\mathbb{D})$

Pf: Exercise.

By chapter III. $\exists\ V_1: \mathbb{D}\to\mathbb{R}$ s.t $f_1 = u_1 + iV_1 \in H(\mathbb{D})$

Let $f = f_1\circ h$. $\Rightarrow$ $f\in H(G)$ and

$$f = (\mathrm{Re} f_1)\circ h + i(\mathrm{Im} f_1)\circ h$$

$$\Rightarrow \quad u = u_1\circ h = \mathrm{Re} f$$

Thus $V = \mathrm{Im} f = \mathrm{Im} f_1\circ h$ is the desired function.

$(j)\Rightarrow(g)$ Suppose $f: G\to\mathbb{C}$ is analytic and $f\neq 0$ in G.

Let $u = \mathrm{Re} f$, $V = \mathrm{Im} f$.

Set $U: G\to\mathbb{R}$ to be

$$U = \log|f(z)|$$

Claim: U is harmonic in G

Pf: Exercise

By (j), $\exists$ a harmonic $V: G\to\mathbb{C}$ s.t

$$\hat{g} = U + iV \in H(G)$$

Set $h = e^{\hat{g}}$ Then $h\in H(G)$ and $h\neq 0$ in G.

Moreover, $\left|\frac{f(z)}{h(z)}\right| = 1$

Thus $\frac{f}{h}\equiv c$ with $|c| = 1$

Note $c = e^{\alpha}$ for some $\alpha\in\mathbb{C}$

$\Rightarrow$ $f = ch = e^{\alpha+\hat{g}} = e^{g}$ where $g = \alpha+\hat{g}$

Example: Let $G = \mathbb{C} - \{z = re^{ir} : 0\le r<\infty\}$

prove G is simply connected

Pf: $\mathbb{C}_\infty - G = \{re^{ir} : 0\le r\le\infty\}$.

Pf: $\mathbb{C}_\infty - G = \{ re^{ir} : 0 \le r \le \infty \}$.

It is connected.

By the above thm, G is simply connected.

3. Mittag-Leffler's Thm

Q: Let $G \subset \mathbb{C}$ open and $\{a_k\} \subset G$ a sequence of distinct pts s.t $\{a_k\}$ has no limit pt in G.

For each $k \ge 1$, given rational functions

$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z-a_k)^j} \quad (*)$$

Where $m_k \in \mathbb{Z}^+$, $A_{jk} \in \mathbb{C}$.

Does there exist $f \in H(G)$ whose poles are exactly $\{a_k\}$ and s.t the singular part of f at a_k is $S_k(z)$.

The answer is yes by Mittag-Leffler's Thm.

Thm (Mittag-Leffler's Thm) Let G be an open set, $\{a_k\}$ a sequence of distinct pts in G without a limit pt in G, and let $\{S_k(z)\}$ be the sequence of rational functions given by $(*)$ Then there is a meromophic function f on G whose poles are exactly the pts $\{a_k\}$ and s.t the singular part of f at a_k is $S_k(z)$

Idea of Pf: Naively, we can sum up $S_k(z)$'s to get $\sum_{k=1}^{\infty} S_k(z)$. But this may not converge.

To overcome this, we will apply Runge's Thm to

To overcome this, we will apply Runge's Thm to find rational functions $\{R_k(z)\}$ with poles in $\mathbb{C}_\infty - G$ s.t $|S_k(z) - R_k(z)|$ small and thus $\sum_{k=1}^{\infty}(S_k(z) - R_k(z))$ converges.

Pf: Step 1: Let $\{K_n\}_{n=1}^{\infty}$ be an exhaustion sequence of G, s.t

$$G = \bigcup_{n=1}^{\infty} K_n, \quad K_n \subset \operatorname{int} K_{n+1}.$$

and each component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$. (see proposition VII.1.2)

As each K_n is compact and $\{a_k\}$ has no limit pt in G,

$\Rightarrow$ There are only finite a_k's in each K_n.

Define

$$I_1 = \{k : a_k \in K_1\}$$

$$I_n = \{k : a_k \in K_n - K_{n-1}\} \text{ for } n \geq 2.$$

Define functions f_n by

$$f_n(z) = \sum_{k \in I_n} S_k(z). \text{ for } n \geq 1.$$

Then f_n is rational and its poles are pts $\{a_k : k \in I_n\} \subset K_n - K_{n-1}$. (If $I_n = \phi$. let $f_n = 0$)

Since f_n has no poles in K_{n-1} (for $n \geq 2$), it is analytic in a nbhd of K_{n-1}.

Step 2: Apply Runge's Thm with $E = \mathbb{C}_\infty - G$, $\exists$ a rational $R_n(z)$ with poles in $\mathbb{C}_\infty - G$ s.t

$$|f_n(z) - R_n(z)| < (\tfrac{1}{2})^n \text{ for } z \in K_{n-1}$$

$R_n(z)$ with poles in $\mathbb{C}_\infty - G$ s.t

$$|f_n(z) - R_n(z)| < \left(\tfrac{1}{2}\right)^n \text{ for } z \in K_{n-1}.$$

Claim: $f(z) = f_1(z) + \sum_{n=2}^{\infty} [f_n(z) - R_n(z)]$ $(**)$

is the desired meromorphic function.

Pf: ① f is analytic in $G - \{a_k : k \geq 1\}$.

Indeed, let $K \subset G - \{a_k : k \geq 1\}$ be compact. Then

$K \subset K_N$ for some large N. $\Rightarrow$

When $n \geq N$, we have $|f_n(z) - R_n(z)| < \left(\tfrac{1}{2}\right)^n$ on K.

Thus $(**)$ converges uniformly on K.

As K is abitrary, $\Rightarrow$ $(**)$ converges in $H(G - \{a_k : k \geq 1\})$.

② a_k is a pole of f.

Fix $k \geq 1$. Choose $r > 0$ s.t $\overline{B(a_k, r)} \subset G$. and

$|a_k - a_j| > R$ for $\forall j \neq k$.

Note $\overline{B(a_k; r)} \subset K_{n_0}$ for some n_0.

Thus $f_n - R_n$ is analytic in $B(a_k; r)$ for large n.

$\Rightarrow$ $f(z) = S_k(z) + g(z)$ for $0 < |z - a_k| < R$

where $g(z)$ is analytic in $B(a_k; r)$

Hence $z = a_k$ is a pole of f and $S_k(z)$ is its singular part. This completes the pf.