2. Simply connectedness

Defn. Let \( X \) and \( \mathbb{S} \) be metric spaces. A homeomorphism between \( X \) and \( \mathbb{S} \) is a continuous map \( f: X \to \mathbb{S} \) which is one-to-one, onto and s.t. \( f^{-1}: \mathbb{S} \to X \) is also continuous.

If such \( f: X \to \mathbb{S} \) exists, we say \( X \) and \( \mathbb{S} \) are homeomorphic.

Ex. Verify \( f: \mathbb{C} \to \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) given by \( f(z) = \frac{z}{1 + |z|^2} \) is a homeomorphism, thus \( \mathbb{C} \) and \( \mathbb{D} \) are homeomorphic.

Ex. Prove all annuli are homeomorphic to the punctured disk.

Thm. Let \( G \) be an open connected subset of \( \mathbb{C} \). Then TFAE:

(a) \( G \) is simply connected;
(b) \( n(\gamma; a) = 0 \) for every closed rectifiable curve \( \gamma \) in \( G \) and every \( a \) at \( \mathbb{C} - G \);
(c) \( \mathbb{C} \setminus G \) is connected;
(d) For any \( f \in \mathbb{H}(G) \), \( \exists \) a sequence of polynomials that converges to \( f \) in \( \mathbb{H}(G) \);
(e) For any \( f \in \mathbb{H}(G) \) and any closed rectifiable curve \( \gamma \) in \( G \), \( \int_{\gamma} f = 0 \);
(f) Every function \( f \in \mathbb{H}(G) \) has a primitive.
(g) For any \( f \in \mathbb{H}(G) \) s.t. \( f(z) \neq 0 \) in \( G \), \( \exists \) \( g \in \mathbb{H}(G) \).
(g) For any \( f \in H(G) \) s.t. \( f(z) \neq 0 \) in \( G \), \( \exists g \in H(G) \) s.t. \( f(z) = e^{g(z)} \).

(h) For any \( f \in H(G) \) s.t. \( f(z) \neq 0 \) in \( G \), \( \exists g \in H(G) \) s.t. \( f(z) = [g(z)]^2 \).

(i) \( G \) is homeomorphic to the unit disk.

(j) If \( u : G \to \mathbb{R} \) is harmonic, then \( \exists \) a harmonic \( v : G \to \mathbb{R} \) s.t. \( f = u + iv \) is analytic on \( G \).

**Proof:** We will prove \((a) \Rightarrow (b) \Rightarrow \ldots \Rightarrow (i) \Rightarrow (a)\) and \((b) \Rightarrow (j) \Rightarrow (g)\).

\((a) \Rightarrow (b)\)

Recall Cauchy's Thm: If \( f \in H(G) \) and \( \gamma \) \( \subset \) \( G \) closed rectifiable s.t. \( \gamma \cap \partial G = 0 \), then

\[ \int_{\gamma} f = 0 \]

As

\[ h(\gamma; a) = \frac{1}{2i\pi} \int_{\gamma} \frac{1}{z-a} \, dz = 0 \]

\[ \frac{1}{z-a} \in H(\gamma) \]

\((b) \Rightarrow (c)\). Suppose \( \infty - G \) is not connected, then

\[ \infty - G = A \cup B \]

where \( A, B \) are disjoint nonempty, closed subset of \( \infty - G \). As \( \infty \in A \cup B \),

WLOG, we assume \( \infty \in B \). Then \( A \) must be a compact subset of \( G \).

But then \( G_1 = G \cup A = \infty - B \) is an open set in \( G \) and contains \( A \).
By proposition 1.1, \( \exists \) a finite number of polygons \( \sigma_1, \ldots, \sigma_m \) in \( G \), \(-A = G \) s.t. for all \( f \in H(G) \)

\[
    f(z) = \frac{m}{2\pi i} \sum_{k=1}^{m} \int_{\sigma_k} \frac{f(w)}{w-z} \, dw
\]

for all \( z \epsilon A \). In particular, if \( f(z) \not\equiv 1 \), then

\[
    1 = \frac{m}{2\pi i} \sum_{k=1}^{m} h(\sigma_k; z) \text{ for } z \epsilon A.
\]

Thus for any \( z \epsilon A \), \( \exists \) at least one polygon \( \sigma_k \) in \( G \) s.t. \( h(\sigma_k; z) \not\equiv 0 \). This contradicts (b).

(c) \( \Rightarrow \) (d) See Corollary 1.5. Recall let \( E = \mathbb{R} \times \mathbb{R} \).

by Range's Thm, \( \exists \) rational \( \{R_n\} \) whose poles in \( E \) s.t. \( R_n \rightarrow f \) in \( H(G) \).

(d) \( \Rightarrow \) (e) Let \( \partial G \) closed rectifiable, \( f \in H(G) \).

Then \( \exists \{R_n\} \) s.t. \( R_n \rightarrow f \) in \( H(G) \).

As \( \int_{\partial G} R_n = 0 \Rightarrow \int_{\partial G} f = 0 \)

(e) \( \Rightarrow \) (f) Fix \( a \epsilon G \). Define

\[
    F(z) = \int_{\gamma} f(w) \text{ where } \gamma \text{ is any rectifiable curve in } G \text{ from } a \text{ to } z.
\]

Then \( F \) is well-defined and \( F' = f \) (see the pf of Corollary IV.6.16).

(f) \( \Rightarrow \) (g) \( (g = \log f \Rightarrow g' = \frac{f'}{f}) \)

As \( f \not\equiv 0 \) in \( G \), \( \Rightarrow \frac{f'}{f} \in H(G) \)

or \( \int_{\partial G} \frac{f'}{f} = 0 \)
As \( f \to \infty \), \( \Rightarrow \frac{1}{f} \in H(\overline{G}) \).

By (f), \( \exists F \in H(G) \) s.t. \( F' = \frac{1}{f} \).

Consider \( h \cdot f e^{-F} \Rightarrow \)

\[
\begin{align*}
h' &= f' e^{-F} - f e^{-F} F' \\
&= e^{-F}(f' - f F') \\
&= 0 \\
\Rightarrow h &\equiv C \Rightarrow f = C e^{F}
\end{align*}
\]

Let \( C = e^{\xi} \), \( \xi \in \mathbb{C} \).

Let \( g = \xi + \overline{F} \Rightarrow f = e^{g} \).

\((g) \Rightarrow (h)\) By \((g)\), \( \exists G \in H(G) \) s.t. \( f = e^{g(2)} \).

Let \( h = e^{\frac{g}{2}} \Rightarrow h^2 = f \).

\((h) \Rightarrow (i)\) If \( G = \mathbb{C} \), then the function \( \mathbb{C}(1+1) \) gives a homeomorphism from \( \mathbb{C} \) to \( \mathbb{D} \).

If \( G \neq \mathbb{C} \), by the pf of Riemann mapping thm, \( \exists f \in H(G) \) one-to-one, onto \( \mathbb{D} \).

\((i) \Rightarrow (a)\) Trivial.

\((h) \Rightarrow (j)\) First recall we have proved \((j)\) for \( G = \mathbb{C} \) or \( \mathbb{D} \).

Now if \( G \neq \mathbb{C} \), then the Riemann mapping thm, \( \Rightarrow \exists h \in H(G) \), s.t. \( h \) is one-to-one and \( h(G) = \mathbb{D} \).

If \( U : G \to \mathbb{R} \) harmonic,

Claim: \( U_i = U \circ h^{-1} \) is harmonic on \( \mathbb{D} \).

Pf: Exercise.

By Riemann \( \exists V_i : D \to \mathbb{R} \) s.t. \( f_i = U_i + iV_i \in H(D) \).
Exercise.

By Chapter III, \( \exists V: D \to \mathbb{R} \) s.t \( f_i = u_i + i v_i \in H(\mathbb{D}) \)

Let \( f = f_i \circ h \). \( \Rightarrow f \in H(G) \) and

\[
\hat{f} = (\text{Re} f_i) \circ h + i (\text{Im} f_i) \circ h
\]

\( \Rightarrow u = u_i \circ h = \text{Re} f \)

Thus \( V = \text{Im} f = \text{Im} f_i \circ h \) is the desired function.

(i) \( \Rightarrow \) (g) Suppose \( f: G \to \mathbb{C} \) is analytic and \( f \neq 0 \) in \( G \)

Let \( U = \text{Re} f \), \( V = \text{Im} f \).

Set \( U: G \to \mathbb{R} \) to be

\[
U = \log |f(z)|
\]

Claim: \( U \) is harmonic in \( G \)

**Pf:** Exercise

By (i), \( \exists a \) harmonic \( V: G \to \mathbb{C} \) s.t

\[
\hat{g} = u + i v \in H(G)
\]

Set \( h = e^\hat{g} \). Then \( h \in H(G) \) and \( h \neq 0 \) in \( G \).

Moreover, \( |\frac{f(z)}{h(z)}| = 1 \)

Thus \( \frac{f}{h} \equiv c \) with \( |c| = 1 \)

Note \( c = e^{\alpha} \) for some \( \alpha \in \mathbb{C} \)

\( \Rightarrow f = c h = e^{\alpha + \hat{g}} = e^{\hat{g}} \) where \( g = \alpha + \hat{g} \)

**Example:**

Let \( G = \mathbb{C} - \{ z = r e^{i \pi} : 0 \leq r < \infty \} \)

prove \( G \) is simply connected.

**Pf:** \( C_{\infty} - G = \{ z : e^{i \pi} : 0 \leq r \leq \infty \} \).
Pf: \( C^\infty - G = \{ x e^{ir} : 0 \leq r \leq \infty \} \).

It is connected.

By the above thm, \( G \) is simply connected.

3. Mittag-Leffler's Thm

Q: Let \( G \subset \mathbb{C} \) open and \( \{a_k\} \subset G \) a sequence of distinct pts s.t \( \{a_k\} \) has no limit pt in \( G \).

For each \( k \geq 1 \), given rational functions
\[
S_k(z) = \frac{\sum_{j=1}^{m_k} A_{jk}}{(z-a_k)^{r}} \tag{x}
\]

Where \( m_k \in \mathbb{Z}^+ \), \( A_{jk} \in \mathbb{C} \).

Does there exist \( f \in H(G) \) whose poles are exactly \( \{a_k\} \) and s.t the singular part of \( f \) at \( a_k \) is \( S_k(z) \).

The answer is yes by Mittag-Leffler's Thm.

Thm (Mittag-Leffler's Thm) Let \( G \) be an open set, \( \{a_k\} \) a sequence of distinct pts in \( G \) without a limit pt in \( G \), and \( \{S_k(z)\} \) be the sequence of rational functions given by \( (x) \) Then there is a meromorphic function \( f \) on \( G \) whose poles are exactly the pts \( \{a_k\} \) and s.t the singular part of \( f \) at \( a_k \) is \( S_k(z) \).

Idea of Pf: Naively, we can sum up \( S_k(z) \)'s to get
\[
\sum_{k=1}^{\infty} S_k(z) \tag{y}
\]

But this may not converge.

To overcome this, we will apply Runge's Thm to
To overcome this, we will apply Runge’s Thm to find rational functions \( \{ R_k(\mathbf{z}) \} \) with poles in \( \mathbb{C}^\infty - \mathbb{G} \), s.t. \( |S_k(\mathbf{z}) - R_k(\mathbf{z})| \) small and thus \( \sum_{k=1}^{\infty} (S_k(\mathbf{z}) - R_k(\mathbf{z})) \) converges.

**Pf:** Step 1: Let \( \{ K_n \}_{n=1}^{\infty} \) be an exhaustion sequence of \( \mathbb{G} \),

\[
G = \bigcup_{n=1}^{\infty} K_n, \quad K_n \subset \text{int } K_{n+1}.
\]

and each component of \( \mathbb{C}^\infty - K_n \) contains a component of \( \mathbb{C}^\infty - \mathbb{G} \). (see proposition VII.1.2)

As each \( K_n \) is compact and \( \{ f_k \} \) has no limit pt in \( \mathbb{G} \),

\[ \Rightarrow \text{There are only finite } f_k \text{'s in each } K_n. \]

Define

\[ I_1 = \{ k : f_k \in K_1 \} \]

\[ I_n = \{ k : f_k \in K_n - K_{n-1} \} \text{ for } n \geq 2. \]

Define functions \( f_n \) by

\[ f_n(\mathbf{z}) = \sum_{k \in I_n} S_k(\mathbf{z}). \text{ for } n \geq 1. \]

Then \( f_n \) is rational and its poles are pts \( \{ k \in I_n \} \subset K_n - K_{n-1}. \) (If \( I_n = \emptyset \), let \( f_n = 0 \))

Since \( f_n \) has no poles in \( K_{n-1} \) (for \( n \geq 2 \)), it is analytic in a nbhd of \( K_{n-1} \).

**Step 2:** Apply Runge’s Thm with \( E = \mathbb{C}^\infty - \mathbb{G} \), \( \exists \text{ a rational } R_n(\mathbf{z}) \) with poles in \( \mathbb{C}^\infty - \mathbb{G} \) s.t.

\[ 0 \leq |S_n(\mathbf{z}) - R_n(\mathbf{z})| \text{ for } \mathbf{z} \in K_{n-1}. \]
$K_n(z)$ with poles in $C \setminus \mathbb{G} > 0$

$|f_n(z) - R_n(z)| < (\frac{1}{2})^n$ for $z \in K_{n-1}$.

Claim: $f(z) = f_1(z) + \sum_{n=2}^{\infty} \left[ f_n(z) - R_n(z) \right]$ (**) is the desired meromorphic function.

Pf: (1) $f$ is analytic in $G - \{a_k: k \geq 1\}$.

Indeed, let $K \subset G - \{a_k: k \geq 1\}$ be compact. Then $K \subset K_N$ for some large $N$. \Rightarrow

When $n \geq N$, we have $|f_n(z) - R_n(z)| < (\frac{1}{2})^n$ on $K$.

Thus (**) converges uniformly on $K$.

As $K$ is arbitrary, \Rightarrow (**) converges in $H(G - \{a_k: k \geq 1\})$.

(2) $a_k$ is a pole of $f$.

Fix $k \geq 1$. Choose $r > 0$ s.t. $\overline{B(a_k, r)} \subset G$ and $|a_k - a_j| > r$ for $\forall j \neq k$.

Note $\overline{B(a_k; r)} \subset K_N$ for some $N$.

Thus $f_n - R_n$ is analytic in $B(a_k; r)$ for large $n$.

\Rightarrow $f(z) = S_k(z) + g(z)$ for $0 < |z - a_k| < r$.

Where $g(z)$ is analytic in $B(a_k; r)$.

Hence $z = a_k$ is a pole of $f$ and $S_k(z)$ is its principal part. This completes the pf.