Solutions: Homework 1

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Problem 1. Prove the following Minimum Principle. If f is a non-constant analytic function on a bounded open set G and is continuous on $\overline{G}$, then either f has a zero in G or $|f|$ assumes its minimum value on ∂G.

Proof. Suppose f has no zero in G. If f has a zero on ∂G, then we are done because that is the minimum. Suppose that f has no zeroes in $\overline{G}$. Then $1/f$ is a non-constant analytic function on G and is continuous on $\overline{G}$. Then, by the second version of the Maximum Modulus Theorem applied to $1/f$, we get

$$\max\left\{\frac{1}{|f(z)|} : z \in \overline{G}\right\} = \max\left\{\frac{1}{|f(z)|} : z \in \partial G\right\}.$$

But this is the same as saying that

$$\min\{|f(z)| : z \in \overline{G}\} = \min\{|f(z)| : z \in \partial G\}.$$

So $|f|$ assumes its minimum value on ∂G.

□

Problem 2. Let G be a bounded region and suppose f is continuous on $\overline{G}$ and analytic on G. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all z on the boundary of G then either f is a constant function or f has a zero in G.

Proof. Suppose f has no zeroes in G. Then by Problem 1 above, we know that $|f(z)| \geq c$ for all $z \in G$, and by the second version of the Maximum Modulus Theorem, we know that $|f(z)| \leq c$ for all $z \in G$. This shows that $|f(z)| = c$ for all $z \in G$. By the first version of the Maximum Modulus Theorem, this implies that f is a constant function. □

Problem 3. Suppose that both f and g are analytic on $\overline{B}(0; R)$ with $|f(z)| = |g(z)|$ for $|z| = R$. Show that if neither f nor g vanishes in $B(0; R)$ then there is a constant $\lambda, |\lambda| = 1$, such that $f = \lambda g$.

Proof. Suppose that f and g has no zeroes on $\partial B(0; R)$. Then f/g is analytic on $\overline{B}(0; R)$ with $|f(z)/g(z)| = 1$ on $\partial B(0; R)$ and with no zeroes in $B(0; R)$. So by Problem 2 above, f/g is a constant function, say, λ with $|\lambda| = 1$. So $f = \lambda g$ with $|\lambda| = 1$. Now suppose that f and g has zeroes on $\partial B(0; R)$. Note that they have the same zeroes. Also, since f and

g are analytic on $\overline{B}(0;R)$, their zeroes are isolated, and hence there are only finitely many of them, say, $z_1, ..., z_n$. Suppose z_1 is a zero of f of multiplicity k and of g with multiplicity k'. Then $f(z) = (z-z_1)^k f_1(z)$ and $g(z) = (z-z_1)^{k'} g_1(z)$ with f_1 and g_1 both analytic on $\overline{B}(0;R)$ and $f_1(z_1) \neq 0$ and $g_1(z) \neq 0$. So we have $|z-z_1|^k |f_1(z)| = |z-z_1|^{k'} |g_1(z)|$ for all $|z| = R$. If $k \neq k'$, this gives a contradiction by taking z (with $|z| = R$) arbitrarily close to z_1. So $k = k'$. Similarly, for any zero in $\partial B(0;R)$, the multiplicities are the same. So, let the multiplicity of z_i be k_i. Then, if we define

$$\tilde{f}(z) = \frac{f(z)}{\prod (z-z_i)_i^k} \text{ and } \tilde{g}(z) = \frac{g(z)}{\prod (z-z_i)_i^k}$$

we get that $\tilde{f}$ and $\tilde{g}$ are analytic and has no zeroes in $\overline{B}(0;R)$ and $|\tilde{f}(z)| = |\tilde{g}(z)|$ for $|z| = R$. So $\tilde{f} = \lambda \tilde{g}$ for some constant λ with $|\lambda| = 1$. Hence $f = \lambda g$. □

Problem 4. Let f be analytic in the disk $B(0;R)$ and for $0 \leq r < R$ define $A(r) = \max\{\text{Re} f(z) : |z| = r\}$. Show that unless f is a constant, $A(r)$ is a strictly increasing function of r.

Proof. Suppose $r < s < R$. Let $g(z) = e^{f(z)}$. Then g is analytic in $B(0;R)$, and $|g(z)| = e^{\text{Re}\, f(z)}$. By the second version of the Maximum Modulus Theorem applied to g and $G = B(0;s)$, we have $\max\{e^{\text{Re}\, f(z)} : |z| = s\} = \max\{e^{\text{Re}\, f(z)} : |z| \leq s\} \geq \max\{e^{\text{Re}\, f(z)} : |z| = r\}$. Since exp is an increasing function, this is the same as saying that $A(s) \geq A(r)$. So $A(.)$ is an increasing function. Now suppose that $A(s) = A(r)$ for some $r < s < R$. Then there exists $z_0 \in |z| = r$ such that $|g(z_0)| \geq |g(z)|$ for all $|z| < s$. By the first version of the Maximum Modulus Theorem, this implies that f must be a constant function. So $A(.)$ has to be a strictly increasing function. □

Problem 5. Does there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$?

Proof. For any analytic function $f : D \to D$, we have

$$\left| f'\left(\frac{1}{2}\right) \right| \leq \frac{1 - |f(\frac{1}{2})|^2}{1 - (\frac{1}{2})^2}.$$

So if $f(\frac{1}{2}) = \frac{3}{4}$, we should have

$$\left| f'\left(\frac{1}{2}\right) \right| \leq \frac{7}{12} < \frac{2}{3}$$

So such an f cannot exist. □

Problem 6. Suppose $f : D \to \mathbb{C}$ satisfies Re $f(z) \geq 0$ for all z in D and suppose that f is analytic and not constant.
(a) Show that Re $f(z) > 0$ for all z in D.
(b) By using an appropriate Möbius transformation, apply Schwarz's Lemma to prove that if $f(0) = 1$ then

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}$$

for $|z| < 1$. What can be said if $f(0) \neq 1$?
(c) Show that if $f(0) = 1$, f also satisfies

$$|f(z)| \geq \frac{1 - |z|}{1 + |z|}.$$

Proof. (a) By the open mapping theorem, $f(D)$ is open in $\mathbb{C}$. Since $f(D) \subset \{z : \text{Re } z \geq 0\}$ and $f(D)$ is open, it should be contained in the interior of $\{z : \text{Re } z \geq 0\}$, which is $\{z : \text{Re } z > 0\}$. Hence $\text{Re } f(z) > 0$ for all z in D.
(b) Let $h(z) = \frac{z-1}{z+1}$. Then h is a Möbius transformation that maps the right half plane to the unit disk. Then $h \circ f : D \to D$ is analytic and not constant. $h \circ f(0) = h(1) = 0$. Hence, by Schwarz's Lemma, we have

$$|(h \circ f)(z)| \leq |z|$$

for all $z \in D$. That is, for all $z \in D$,

$$\frac{|f(z) - 1|}{|1 + f(z)|} \leq |z|.$$

Hence

$$|f(z) - 1| \leq |z|(|1 + f(z)|) \leq |z|(1 + |f(z)|)$$

But we have

$$|f(z)| - 1 \leq |f(z) - 1| \leq |z|(1 + |f(z)|)$$

This proves the inequality.
If $f(0) = a \neq 1$, then $(h \circ f)(0) = h(a) \neq 0$. If $g = \varphi_{h(a)} \circ h \circ f$, then we can apply Schwarz's Lemma on g to get $|g(z)| \leq |z|$ for all $z \in D$. A calculation shows that

$$g(z) = \frac{\overline{a} + 1}{a + 1} \frac{f(z) - a}{f(z) + \overline{a}}.$$

Then

$$|g(z)| = \frac{|f(z) - a|}{|f(z) + \overline{a}|}.$$

So we have

$$|f(z) - a| \leq |z|(|f(z) + \overline{a}|)$$

Using the triangle inequality, we get

$$|f(z)| - |a| \leq |z|(|f(z)| + |a|)$$

which implies, for all $z \in D$

$$|f(z)| \leq |a| \left(\frac{1 + |z|}{1 - |z|} \right) = |f(0)| \left(\frac{1 + |z|}{1 - |z|} \right).$$

(c) By part (a), f cannot have a zero in D. So $1/f$ is analytic and non constant and Re $1/f(z) > 0$ for all $z \in D$. Also $(1/f)(0) = 1$. hence we can apply part (b) to $1/f$ to get

$$\frac{1}{|f(z)|} \leq \frac{1 + |z|}{1 - |z|}$$

This proves part (c). $\square$

Problem 7. Suppose f is analytic in some region containing $\overline{B}(0;1)$ and $|f(z)| = 1$ where $|z| = 1$. Find a formula for f.

Proof. Suppose f has no zeroes in D. Then by Problem 3 above, applied to $g \equiv 1, f = \lambda$ for some $|\lambda| = 1$. Suppose that f is not constant and hence by the Maximum Modulus theorem f maps D to D. Also, f has only finitely many zeros in $\overline{B}(0;1)$. Now suppose that f has zeroes $z_1, ..., z_n$ in D of multiplicity $k_1, ..., k_n$ respectively. The function $(\varphi_{z_i})^{k_i}$ has a zero only at z_i and of multiplicity k_i. Let $\tilde{f}$ denote the function

$$\tilde{f} = \frac{f}{\prod(\varphi_{z_i})^{k_i}}.$$

Then $\tilde{f}$ is analytic in some region containing $\overline{B}(0;1)$ and since φ_{z_i} maps ∂D to $\partial D, |\tilde{f}(z)| = 1$ for $|z| = 1$. Also, $\tilde{f}$ has no zeroes in D. Therefore $\tilde{f}$ has to be a constant function. Hence

$$f = \lambda \prod(\varphi_{z_i})^{k_i}$$

for some $|\lambda| = 1$. □

Problem 8. Is there an analytic function f on $B(0;1)$ such that $|f(z)| < 1$ for $|z| < 1, f(0) = \frac{1}{2}$, and $f'(0) = \frac{3}{4}$? If so, find such an f. Is it unique?

Proof. Suppose such an f exists. Let $\tilde{f} = \varphi_{1/2} \circ f$. Then $\tilde{f}(0) = 0$. Also $|\tilde{f}(z)| \leq 1$ for $z \in D. \tilde{f}'(0) = (\varphi_{1/2})'(f(0))f'(0) = \frac{3}{4}(1 - (\frac{1}{2})^2)^{-1} = 1$. By Schwarz's Lemma, $\tilde{f}(z) = cz$ for all $z \in D$ for some c with $|c| = 1$. So $f(z) = \varphi_{-1/2}(cz)$. Putting back the condition $f'(0) = \frac{3}{4}$, we see that c has to be 1. So $f = \varphi_{-1/2}$. Conversely, it is obvious that $\varphi_{-1/2}$ satisfies all the conditions given in the problem. So, such an f exists and is unique. □