Solutions: Homework 2

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Problem 1. Let $f(z) = \frac{1}{z(z-1)(z-2)}$; give the Laurent Expansion of $f(z)$ in each of the following annuli: (a) ann $(0; 1, 2)$; (b) ann $(0; 2, \infty)$.

Proof.

$$f(z) = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

(a) If $1 < |z| < 2$, $\frac{1}{|z|} < 1$, hence

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \sum_{n=1}^{\infty} \frac{1}{z^n} = \sum_{n=-\infty}^{-1} z^n$$

and $|\frac{z}{2}| < 1$, hence

$$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})} = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} z^n$$

So, $f(z) = \sum a_n z^n$ where

$$a_n = \begin{cases} -1 & \text{if } n \leq -2 \\ -\frac{1}{2^{n+2}} & \text{if } n \geq -1. \end{cases}$$

(b) If $|z| > 2$, as above, we have

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \sum_{n=1}^{\infty} \frac{1}{z^n} = \sum_{n=-\infty}^{-1} z^n$$

but $|\frac{2}{z}| < 1$, hence

$$\frac{1}{z-2} = \frac{1}{z(1-\frac{2}{z})} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=-\infty}^{-1} \frac{1}{2^{n+1}} z^n$$

So,

$$f(z) = \sum_{n=-\infty}^{-2} (2^{-(n+2)} - 1) z^n$$

□

Problem 2. Show that $f(z) = \tan z$ is analytic in $\mathbb{C}$ except for simple poles at $z = \frac{\pi}{2} + n\pi$, for each integer n. Determine the singular part of f at each of these poles.

Proof. $\tan z = \frac{\sin z}{\cos z}$. $\sin z$ and $\cos z$ are entire functions, so f should be analytic in $\mathbb{C}$ except possibly when $\cos z = 0$, which happens iff $z = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$. Also, note that $\sin(\frac{\pi}{2} + n\pi) = \pm 1 \neq 0$, so f certainly has a pole at $\frac{\pi}{2} + n\pi$. Now note that all the zeroes of $\cos z$ are simple, and hence all the poles of f are also simple. So, the singular part of f at any of these poles will be of the form $\frac{c_n}{z - (\frac{\pi}{2} + n\pi)}$, where

$$c_n = \lim_{z \to \frac{\pi}{2} + n\pi} \frac{(z - (\frac{\pi}{2} + n\pi))(\sin z)}{\cos z} = \frac{\sin(\frac{\pi}{2} + n\pi)}{(\cos)'(\frac{\pi}{2} + n\pi)} = -1.$$

So, the singular part at $\frac{\pi}{2} + n\pi$ is $\frac{-1}{z - (\frac{\pi}{2} + n\pi)}$. □

Problem 3. If $f : G \to \mathbb{C}$ is analytic except for poles show that the poles of f cannot have a limit point in G.

Proof. Let a be a limit point of the set of poles of f. Since poles are isolated singularities and f is analytic except for poles, f is analytic at a. Hence f is well-defined in a small neighbourhood of a. This contradicts the fact that a is a limit point of the set of poles of f. So, the poles of f cannot have a limit point in G. □

Problem 4. Suppose that f has an essential singularity at $z = a$. Prove the following strengthened version of the Casorati-Weierstrass Theorem. If $c \in \mathbb{C}$ and $\epsilon > 0$ are given then for each $\delta > 0$ there is a number α, $|c - \alpha| < \epsilon$, such that $f(z) = \alpha$ has infinitely many solutions in $B(a; \delta)$.

Proof. Let $c \in \mathbb{C}$ and $\epsilon > 0$ be fixed. Let $G_n = f(B(a; 1/n) \setminus \{a\})$ for $n \geq 1$. Then G_n is open in $\mathbb{C}$, by the Open Mapping Theorem, and G_n is dense in $\mathbb{C}$ by the Casorati-Weierstrass theorem. By the Baire Category Theorem, $\cap_{n=1}^{\infty} G_n$ is dense in $\mathbb{C}$. Then, $B(c; \epsilon) \cap (\cap_{n=1}^{\infty} G_n) \neq \emptyset$. Let α be such that $\alpha \in B(c; \epsilon) \cap (\cap_{n=1}^{\infty} G_n)$. Then $|c - \alpha| < \epsilon$ and there exists $z_n \in B(a; 1/n) \setminus \{a\}$ such that $f(z_n) = \alpha$. So, for any $\delta > 0$, for $n > 1/\delta$, there exists z_n such that $f(z_n) = \alpha$. Also, note that, $z_n \to a$, but $z_n \neq a$ for all n, and hence $|\{z_n\}| = \infty$. □

Problem 5. Let f be analytic in $G = \{z : 0 < |z - a| < r\}$ except that there is a sequence of poles $\{a_n\}$ in G with $a_n \to a$. Show that for any ω in $\mathbb{C}$ there is a sequence $\{z_n\}$ in G with $a = \lim z_n$ and $\omega = \lim f(z_n)$.

Proof. Suppose that there exists $\omega \in \mathbb{C}$ such that there does not exist any sequence $\{z_n\} \in G$ with $a = \lim z_n$ and $\omega = \lim f(z_n)$. Then there exists $\epsilon > 0, r > \delta > 0$ such that $B(a; \delta) \cap f^{-1}(B(\omega; \epsilon)) = \emptyset$. Otherwise, for all $n >> 0$, there exists $z_n \in B(a; 1/n) \cap f^{-1}(B(\omega; 1/n))$, which means that $z_n \to a$ and $f(z_n) \to \omega$. Define $g : B(a; \delta) \setminus \{a\} \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z) - \omega}$$

and $g(a_n) = 0$ for any $|a_n - a| < \delta$. Clearly, g is analytic at all points other than the a_n's, because $f(z) \neq w$ for all $z \in B(a;\delta)$. At a_n, f is a pole of order, say, k_n. Then $f = f_n/(z-a_n)^{k_n}$ for some f_n analytic around a_n with $f_n(a) \neq 0$ and hence $g(z) = \frac{(z-a_n)^{k_n}}{f_n(z)-\omega(z-a_n)^{k_n}}$ and since $f_n(a_n) \neq 0$, putting $g(a_n) = 0$ makes g analytic around a_n. So, g is analytic in $B(a;\delta) \setminus \{a\}$. So, g has either a removable singularity, a pole or an essential singularity at a. Clearly, it cannot be a pole, as $\lim |g(a_n)| = 0 \neq \infty$. Also, for all $z \in B(a;\delta) \setminus \{a, a_n : n \in \mathbb{N}\}$, $|f(z) - \omega| \geq \epsilon$, and hence $|g(z)| \leq 1/\epsilon$ for $z \in B(a;\delta) \setminus \{a\}$. By the Casorati-Weierstrass theorem, this implies that g cannot have an essential singularity at a. So, g has a removable singularity at a. So, we can extend g to $B(a;\delta)$. Of course, $g(a) = \lim g(a_n) = 0$. But this implies that the zeroes of g has a limit point in $B(a;\delta)$, which implies $g \equiv 0$ in $B(a;\delta)$, which is impossible. This contradicts our initial assumption, and hence completes the proof. □

Problem 6. Calculate the following integrals:

(a)

$$\int_0^\infty \frac{x^2 dx}{x^4 + x^2 + 1}$$

(b)

$$\int_0^\infty \frac{\cos x - 1}{x^2} dx$$

Proof. (a) Let $R > 1$. Let C_R denote the upper semi-circle with center at 0 and radius R, oriented in the counter-clockwise direction. The function $f(z) = \frac{z^2}{z^4+z^2+1}$ has two simple poles at $e^{\pi i/3}$ and $e^{2\pi i/3}$ in the upper half-plane. By the Residue Theorem,

$$\int_{C_R} \frac{z^2}{z^4 + z^2 + 1} dz = 2\pi i (Res_{z=e^{\pi i/3}} f(z) + Res_{z=e^{2\pi i/3}} f(z))$$

$$= 2\pi i \left(\lim_{z \to e^{\pi i/3}} \frac{z^2(z - e^{\pi i/3})}{z^4 + z^2 + 1} + \lim_{z \to e^{2\pi i/3}} \frac{z^2(z - e^{2\pi i/3})}{z^4 + z^2 + 1} \right)$$

$$= 2\pi i \left(\frac{1 + i\sqrt{3}}{4i\sqrt{3}} + \frac{1 - i\sqrt{3}}{4i\sqrt{3}} \right) = \frac{\pi}{\sqrt{3}}$$

Now, splitting C_R into the arc part ($z = Re^{i\theta}$ with $0 < \theta < \pi$) and the x-axis part ($-R < x < R$), we have

$$\int_{C_R} \frac{z^2}{z^4 + z^2 + 1} dz = \int_0^\pi \frac{(Re^{i\theta})^2}{(Re^{i\theta})^4 + (Re^{i\theta})^2 + 1} iRe^{i\theta} d\theta + \int_{-R}^R \frac{x^2}{x^4 + x^2 + 1} dx$$

Since $\frac{x^2}{x^4+x^2+1}$ is an even function, we have

$$\int_{C_R} \frac{z^2}{z^4 + z^2 + 1} dz = \int_0^\pi \frac{i(Re^{i\theta})^3}{(Re^{i\theta})^4 + (Re^{i\theta})^2 + 1} d\theta + 2\int_0^R \frac{x^2}{x^4 + x^2 + 1} dx$$

Let

$$I_R := \int_0^\pi \frac{i(Re^{i\theta})^3}{(Re^{i\theta})^4 + (Re^{i\theta})^2 + 1} d\theta$$

Then

$$|I_R| \le \int_0^{\pi} \frac{R^3}{|(Re^{i\theta})^4 + (Re^{i\theta})^2 + 1|} d\theta \le \int_0^{\pi} \frac{R^3}{R^4 - R^2 - 1} d\theta$$

Note that the second inequality happens only when $R^4 > R^2 + 1$, but we are concerned only with $R >> 1$, so this is fine. So, we have

$$|I_R| \le \frac{\pi R^3}{R^4 - R^2 - 1}$$

for $R >> 1$. This implies that $I_R \to 0$ as $R \to \infty$. By the Residue Theorem calculations, we have

$$\frac{\pi}{\sqrt{3}} = \lim_{R\to\infty} I_R + 2\int_0^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx$$

So, we have

$$\int_0^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx = \frac{\pi}{2\sqrt{3}}.$$

(b) The function $\frac{e^{iz}-1}{z^2}$ has a simple pole at $z = 0$. If $0 < r < R$, let γ be the closed curve depicted in Example 2.7. (Two semicircles in the upper half plane with radii r and R joined at the x-axis.) From Cauchy's theorem, we have $\int_\gamma f = 0$. Breaking γ into its pieces,

$$0 = \int_r^R \frac{e^{ix} - 1}{x^2} dx + \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz + \int_{-R}^{-r} \frac{e^{ix} - 1}{x^2} dx - \int_{\gamma_r} \frac{e^{iz} - 1}{z^2} dz$$

where γ_R and γ_r are the semicircles from R to $-R$ and r to $-r$ respectively, with both oriented in the anti-clockwise direction. By a change of variables, we see that

$$\int_{-R}^{-r} \frac{e^{ix} - 1}{x^2} dx = \int_r^R \frac{e^{-ix} - 1}{x^2} dx$$

So, we have

$$\begin{aligned} 0 &= \int_r^R \frac{e^{ix} + e^{-ix} - 2}{x^2} dx + \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz - \int_{\gamma_r} \frac{e^{iz} - 1}{z^2} dz \\ &= 2\int_r^R \frac{\cos x - 1}{x^2} dx + \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz - \int_{\gamma_r} \frac{e^{iz} - 1}{z^2} dz \end{aligned}$$

Also

$$\left| \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz \right| = \left| i \int_0^{\pi} \frac{e^{iRe^{i\theta}} - 1}{Re^{i\theta}} d\theta \right| \le \frac{1}{R} \int_0^{\pi} |e^{iRe^{i\theta}} - 1| d\theta$$

Note that $|e^{iRe^{i\theta}} - 1| = |e^{-R\sin\theta} e^{iR\cos\theta} - 1| \le e^{-R\sin\theta} + 1$ by the triangle inequality. Since $0 \le \theta\pi, \sin\theta > 0$, and so $e^{-R\sin\theta} \le 1$. Hence, we have

$$\left| \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz \right| \le \frac{2}{R}.$$

Hence

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz} - 1}{z^2} dz = 0$$

Now,

$$\int_{\gamma_r} \frac{e^{iz}-1}{z^2}dz = \int_{\gamma_r} \frac{\cos z - 1}{z^2}dz + i\int_{\gamma_r} \frac{\sin z}{z^2}dz$$

Note that $\frac{\cos z-1}{z^2}$ has a removable singularity at $z=0$, and hence there is a constant $M>0$ such that $\left|\frac{\cos z-1}{z^2}\right| \leq M$ for $|z|\leq 1$. Hence,

$$\left|\int_{\gamma_r} \frac{\cos z - 1}{z^2}dz\right| \leq \pi r M$$

that is

$$\lim_{r\to 0}\int_{\gamma_r} \frac{\cos z - 1}{z^2}dz = 0.$$

Now, $\frac{\sin z}{z^2}$ has a simple pole at 0. So,

$$\int_{C_r} \frac{\sin z}{z^2}dz = 2\pi i$$

where C_r denotes the circle around 0 or radius r. So, we have

$$\int_0^{2\pi} \frac{\sin(re^{i\theta})}{re^{i\theta}}d\theta = 2\pi$$

Splitting into two parts, we have

$$\int_0^{\pi} \frac{\sin(re^{i\theta})}{re^{i\theta}}d\theta + \int_{\pi}^{2\pi} \frac{\sin(re^{i\theta})}{re^{i\theta}}d\theta = 2\pi$$

Changing θ to $\theta+\pi$ in the second integral, we get

$$\int_0^{\pi} \frac{\sin(re^{i\theta})}{re^{i\theta}}d\theta + \int_0^{\pi} \frac{\sin(re^{i\theta+i\pi})}{re^{i\theta+i\pi}}d\theta = 2\pi$$

Simplifying this, we get

$$\int_0^{\pi} \frac{\sin(re^{i\theta})}{re^{i\theta}}d\theta = \pi.$$

But note that

$$\int_{\gamma_r} \frac{\sin z}{z^2}dz = i\int_0^{\pi} \frac{\sin(re^{i\theta})}{re^{i\theta}}d\theta = \pi i$$

So, we have

$$\lim_{r\to 0}\int_{\gamma_r} \frac{e^{iz}-1}{z^2}dz = -\pi$$

So, taking limits as $r\to 0$ and $R\to\infty$ above, we get

$$\int_0^{\infty} \frac{\cos x - 1}{x^2}dx = -\frac{\pi}{2}$$

$\square$

Problem 7. Verify the following equations:
(a)

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \frac{\pi}{2[a(a+1)]^{\frac{1}{2}}}, \text{ if } a > 0;$$

(b)

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin a\pi} \text{ if } 0 < a < 1.$$

Proof. (a) $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. So,

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = 2\int_0^{\pi/2} \frac{d\theta}{2a+1-\cos 2\theta}$$

Putting $2\theta = \pi - \alpha$, we get

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \int_0^{\pi} \frac{d\alpha}{2a+1+\cos\alpha}$$

Since $a > 0, 2a+1 > 1$, and by Example 2.9 in the textbook, we have

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \int_0^{\pi} \frac{d\alpha}{2a+1+\cos\alpha} = \frac{\pi}{\sqrt{(2a+1)^2-1}} = \frac{\pi}{2[a(a+1)]^{\frac{1}{2}}}.$$

(b) Putting $e^x = t$, we have

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \int_0^{\infty} \frac{t^{a-1}}{1+t} dt$$

Since $0 < a < 1$, $0 < 1-a < 1$, and by Example 2.12 in the textbook, we have

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \int_0^{\infty} \frac{t^{-(1-a)}}{1+t} dt = \frac{\pi}{\sin\pi(1-a)} = \frac{\pi}{\sin a\pi}.$$

□

Problem 8. Suppose that f has a simple pole at $z = a$ and let g be analytic in an open set containing a. Show that $Res(fg; a) = g(a)Res(f; a)$.

Proof. Since f has a simple pole at $z = a$ and g is analytic around a, fg either has a simple pole at a or has a removable singularity at a. So, $Res(fg; a) = \lim_{z\to a}(z-a)f(z)g(z) = \lim_{z\to a}(z-a)f(z)\lim_{z\to a} g(z)$ as both limits exist. Hence

$$Res(fg; a) = \lim_{z\to a}(z-a)f(z)\lim_{z\to a} g(z) = g(a)Res(f; a).$$

□