Solutions: Homework 3

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Problem 1. Suppose f is analytic on $\overline{B}(0;1)$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions (counting multiplicities) of the equation $f(z) = z^n$ where n is an integer larger than or equal to 1.

Proof. Let $g(z) = f(z) - z^n$ for all $z \in \overline{B}(0;1)$, and let $h(z) = z^n$. Then since $|f(z)| < 1$ for $|z| = 1$, g has no zeroes or poles on the unit circle. Obviously, h also has no poles or zeroes on the unit circle. Note that, for $|z| = 1$,

$$|g(z) + h(z)| = |f(z)| < 1 = |z^n| < |h(z)| + |g(z)|$$

Then, by Rouché's theorem, $Z_g - P_g = Z_h - P_h$. Since g and h are analytic on $\overline{B}(0;1)$, $P_g = P_h = 0$. Hence $Z_g = Z_h = n$, since h has a zero of multiplicity n at 0 and has no other zeroes. So, the equation $f(z) = z^n$ has n solutions, counting multiplicities. □

Problem 2. Let f be analytic in a neighbourhood of $D = \overline{B}(0;1)$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \leq 1$ for $|z| = 1$, what can you say?

Proof. By the problem above, the equation $f(z) = z$ has exactly one solution in $B(0;1)$. Now, suppose that $|f(z)| \leq 1$ for $|z| = 1$. Suppose that there does not exist z with $|z| = 1$ and $f(z) = z$. Then, applying the Rouché's theorem to $f(z) - z$ and z as in the previous problem, we see that $f(z) = z$ still has a unique solution in $B(0;1)$. If $f(0) = 0$, and if there exists $z \in B(0;1) \setminus \{0\}$ such that $f(z) = z$, then by Schwarz lemma, we know that $f(z) = z$ for all $z \in B(0;1)$. Otherwise, we cannot conclude anything for sure. For example, if $f(z) = z^n$, then it has a unique solution in $B(0;1)$, but if $f(z) = (z^2 + 1)/2$, then it has no solution in $B(0;1)$. □

Problem 3. Prove the following Lemma: If (S, d) is a metric space then

$$\mu(s,t) = \frac{d(s,t)}{1 + d(s,t)}$$

is also a metric on S. A set is open in (S, d) iff it is open in (S, μ); a sequence is a Cauchy sequence in (S, d) iff it is a Cauchy sequence in (S, μ).

Proof. $\mu(s,t) = 0$ iff $d(s,t) = 0$ which happens iff $s = t$. Clearly, $\mu(s,t) = \mu(t,s)$ and $\mu(s,t) \geq 0$ for all $s,t \in S$. Let $s_1, s_2, s_3 \in S$. Let $a = d(s_1,s_2), b = d(s_1,s_3)$ and $c = d(s_3,s_2)$. Then $b \leq a+c \leq a+c+ac$, where the first inequality is due to the triangle inequality applied to d. So $1+b \leq 1+a+c+ac = (1+a)(1+c)$. Again by the triangle inequality, we have $a \leq b+c$, so $a-c \leq b$. Combining there two inequalities, we have $(a-c)(1+b) \leq b(1+a)(1+c)$, which upon rearranging gives,

$$\frac{a}{1+a} - \frac{c}{1+c} \leq \frac{b}{1+b}$$

Putting back the values of a,b,c in terms of $d(.,.)$, we have

$$\mu(s_1,s_2) \leq \mu(s_1,s_3) + \mu(s_3,s_2)$$

for all $s_1,s_2,s_3 \in S$. This proves the triangle inequality. Hence, μ is a metric on S.
Now, let U be open in (S,d). Let $x \in U$. Then there exists $\epsilon > 0$ such that $B_d(x;\epsilon) \subset U$. So, for all $y \in S$ such that $d(y,x) < \epsilon, y \in U$. Now, note that if $\mu(y,x) = \frac{d(y,x)}{1+d(y,x)} < \frac{\epsilon}{1+\epsilon}$, then $d(y,x) < \epsilon$. So, for all $y \in S$ such that $\mu(y,x) < \epsilon/(1+\epsilon), y \in U$. Since $x \in U$ was arbitrary, U is open in (S,μ). Now, let V be open in (S,μ). Let $x \in U$. Then there exists $1 > \epsilon > 0$ such that $B_\mu(x;\epsilon) \subset V$. So, for all $y \in S$ such that $\mu(y,x) < \epsilon, y \in V$. Now, note that if $d(y,x) = \frac{\mu(y,x)}{1-\mu(y,x)} < \frac{\epsilon}{1-\epsilon}$, then $\mu(y,x) < \epsilon$. So, for all $y \in S$ such that $d(y,x) < \epsilon/(1-\epsilon), y \in V$. Since $x \in V$ was arbitrary, V is open in (S,d).
Now, let us define the function $i : (S,d) \to (S,\mu)$ where $i(x) = x$ for all $x \in S$. Then $\mu(i(x),i(y)) \leq d(x,y)$ for all $x,y \in S$, and hence i is Lipschitz, and so uniformly continuous. This implies that if $\{x_n\}$ is a Cauchy sequence in (S,d), it is still a Cauchy sequence in (S,μ). Now, let $\{x_n\}$ be a Cauchy sequence in (S,μ). Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\mu(x_n,x_m) < \frac{\epsilon}{1+\epsilon}$ for all $n,m \geq N$. But this implies that $d(x_n,x_m) < \epsilon$ for all $n,m \geq N$. So $\{x_n\}$ is Cauchy in (S,d). □

Problem 4. Suppose $\{f_n\}$ is a sequence in $C(G,\Omega)$ which converges to f and $\{z_n\}$ is a sequence in G which converges to a point z in G. Show $\lim f_n(z_n) = f(z)$.

Proof. Since f_n converges to f in $C(G,\Omega)$, f_n converges to f uniformly on any compact subset of G. Let $K = \{z_n | n \in \mathbb{N}\} \cup \{z\}$. Then K is compact. Hence $f_n \to f$ uniformly on K. Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $d(f_n(y), f(y)) < \epsilon/2$ for all $n \geq N_1$ and for all $y \in K$. Since f is continuous, there exists $N_2 \in \mathbb{N}$ such that $d(f(z_n), f(z)) < \epsilon/2$ for all $n \geq N_2$. Then, for all $n \geq \max(N_1,N_2)$,

$$d(f_n(z_n), f(z)) \leq d(f_n(z_n), f(z_n)) + d(f(z_n), f(z)) < \epsilon$$

So, $\lim f_n(z_n) = f(z)$. □

Problem 5. (Dini's Theorem) Consider $C(G,\mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G,\mathbb{R})$ which is monotonically increasing (i.e., $f_n(z) \leq f_{n+1}(z)$ for all z in G) and $\lim f_n(z) = f(z)$ for all z in G where $f \in C(G,\mathbb{R})$. Show that $f_n \to f$.

Proof. Replacing f_n with $f - f_n$, we can assume WLOG that $f \equiv 0$ and $\{f_n\}$ is monotonically decreasing. Let $K \subset G$ be compact. Also, note that $f_n \geq 0$ for all $n \geq 1$ and $\lim f_n(z) = 0$ for all $z \in G$. Fix $\epsilon > 0$. Let $z_0 \in K$. Choose $N(z_0) \in \mathbb{N}$ such that $f_n(z_0) < \epsilon/2$ for all $n \geq N(z_0)$. Choose $\delta(z_0) > 0$ such that $|f_{N(z_0)}(z) - f_{N(z_0)}(z_0)| < \epsilon/2$ for all $z \in B(z_0; \delta(z_0))$. Then, we have $f_{N(z_0)}(z) < \epsilon$ for all $z \in B(z_0; \delta(z_0))$. Since the f_n's are monotonically decreasing, this implies that $f_n(z) < \epsilon$ for all $z \in B(z_0; \delta(z_0))$ and for all $n \geq N(z_0)$. Since K is compact, there exists finitely many z_i's, say, for $1 \leq i \leq r$ such that $K \subset \cup_{i=1}^{r} B(z_i; \delta(z_i))$. Let $N = \max\{N(z_i) : 1 \leq i \leq r\}$. Then, for all $n \geq N$, $f_n(z) < \epsilon$ for all $z \in K$. So $f_n \to 0$ uniformly in K. Since K is an arbitrary compact subset in G, this implies that $f_n \to 0$ in $C(G, \mathbb{R})$. $\square$

Problem 6. (a) Let f be analytic on $B(0; R)$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$. If $f_n(z) = \sum_{k=0}^{n} a_k z^k$, show that $f_n \to f$ in $C(G; \mathbb{C})$.
(b) Let $G = \operatorname{ann}(0; 0, R)$ and let f be analytic on G with Laurent series development $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. Put $f_n(z) = \sum_{k=-\infty}^{n} a_k z^k$ and show that $f_n \to f$ in $C(G; \mathbb{C})$.

Proof. (a) Let $K \subset B(0; R)$ be compact. Then there exists $r < R$ such that $K \subset \overline{B}(0; r)$. Since r is less than the radius of convergence of this power series, by Theorem 1.3(c) of Chapter III, we see that $f_n \to f$ uniformly on $\overline{B}(0; r)$, and hence obviously on K. Since K is an arbitrary compact subset of G, this implies that $f_n \to f$ in $C(G; \mathbb{C})$.
(b) Let $K \subset \operatorname{ann}(0; 0, R)$ be compact. Then there exists $0 < r_1 < r_2 < R$ such that $K \subset \overline{\operatorname{ann}}(0; r_1, r_2)$. Let $f_{n,+}(z) = \sum_{k=0}^{n} a_k z^k$. Let $f_+(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_-(z) = \sum_{n=1}^{\infty} a_{-n} z^{-n}$. Then, by Theorem 1.11 in Chapter V, $f_{n,+} \to f_+$ uniformly on $\overline{\operatorname{ann}}(0; r_1, r_2)$, and hence in K. Note that $f_n = f_- + f_{n,+}$. So $f_n \to f_- + f_+ = f$ uniformly on K. Since K is an arbitrary compact subset of G, this implies that $f_n \to f$ in $C(G; \mathbb{C})$.

$\square$