Solutions: Homework 4

February 10, 2020

Problem 1. Prove Vitali's Theorem: If G is a region and $\{f_n\} \in H(G)$ is locally bounded and $f \in H(G)$ that has the property that $A = \{z \in G : \lim f_n(z) = f(z)\}$ has a limit point in G then $f_n \to f$.

Proof. Suppose that $\{f_n\}$ is locally bounded. By Montel's theorem, $\{f_n\}$ is normal. Now suppose that $f_n \not\to f$. Then there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that $\rho(f_{n_k}, f) > \epsilon$ for all n_k. Since $\{f_n\}$ is normal, there exists a subsequence $\{f_{n_{k_l}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_l}} \to g$, as $n_{k_l} \to \infty$ for some $g \in H(G)$. So, $\lim f_{n_{k_l}}(z) = g(z)$ for all $z \in G$. In particular, $g(z) = f(z)$ for all $z \in A$. Since A has a limit point in G, this implies that $f \equiv g$. So, $f_{n_{k_l}} \to f$ in $H(G)$. This contradicts the fact that $\rho(f_{n_{k_l}}, f) > \epsilon$ for all n_{k_l}. So, $f_n \to f$ in H(G). $\square$

Problem 2. Show that for a set $\mathscr{F} \subset H(G)$ the following are equivalent:
(a) $\mathscr{F}$ is normal;
(b) For every $\epsilon > 0$ there is a number $c > 0$ such that $\{cf : f \in \mathscr{F}\} \subset B(0; \epsilon)$ (here $B(0; \epsilon)$ is the ball in $H(G)$ with center at 0 and radius ϵ).

Proof. (a) $\implies$ (b): Suppose $\mathscr{F}$ is normal. Then, by Montel's theorem, it is locally bounded. Let $G = \cup_{n=1}^{\infty} K_n$ where K_n's are compact, and $K_n \subset (K_{n+1})^o$. Let ρ_n and ρ be the metric on $C(G, \mathbb{C})$ associated to these compact subsets of G. Then $\rho_n(f, 0) = \sup\{|f(z)| : z \in K_n\}$ and $\rho(f, 0) = \sum_{n=1}^{\infty} (\frac{1}{2})^n \frac{\rho_n(f,0)}{1+\rho_n(f,0)}$. Since $\mathscr{F}$ is locally bounded, by Lemma 2.8, there exists $M_n \geq 0$ such that $|f(z)| \leq M_n$ for all $f \in \mathscr{F}$ and $z \in K_n$. Now, let $c > 0$. Then $\rho_n(cf, 0) \leq cM_n$ for all $f \in \mathscr{F}$. Let $\epsilon > 0$ be fixed.

$$\rho(cf, 0) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, 0)}{1 + \rho_n(f, 0)} \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{cM_n}{1 + cM_n}$$

Let $N \in \mathbb{N}$ be such that $\frac{1}{2^N} = \sum_{n=N+1}^{\infty} (\frac{1}{2})^n \leq \frac{\epsilon}{2}$. Choose $c > 0$ such that

$$c < \frac{\epsilon}{2(\sum_{n=1}^{N} \frac{M_n}{2^n})}$$

Then, for all $f \in \mathscr{F}$, we have

$$\rho(cf, 0) \leq \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n \frac{cM_n}{1 + cM_n} + \sum_{n=N+1}^{\infty} \left(\frac{1}{2}\right)^n \frac{cM_n}{1 + cM_n} \leq c \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n M_n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n < \epsilon$$

This proves (b).
(b) $\implies$ (a): Conversely, suppose (b) holds. Let $K \subset G$ be a compact set. Let K_n's be as above. Since $(K_n)^o$'s form an open cover of G, there exists $N \in \mathbb{N}$ such that $K \subset K_N$. By (b), there exists $c > 0$ such that $\rho(cf, 0) < \frac{1}{2^{N+1}}$ for all $f \in \mathscr{F}$. Now,

$$\left(\frac{1}{2}\right)^N \frac{\rho_N(cf,0)}{1+\rho_N(cf,0)} \leq \rho(cf,0) < \frac{1}{2^{N+1}}$$

for all $f \in \mathscr{F}$. This implies that $\rho_N(cf,0) < 1$ for all $f \in \mathscr{F}$. This implies that $\sup\{|cf(z)| : z \in K_N\} \leq \sup\{|cf(z)| : z \in K_N\} < 1$, and hence $|f(z)| \leq 1/c$ for all $f \in \mathscr{F}$ and $z \in K$. Since K is an arbitrary compact subset of G, by Lemma 2.8, this proves that $\mathscr{F}$ is locally bounded, and hence normal, by Montel's theorem. This proves (a). □

Problem 3. Let $D = B(0;1)$ and for $0 < r < 1$ let $\gamma_r(t) = re^{2\pi i t}, 0 \leq t \leq 1$. Show that a sequence $\{f_n\}$ in $H(D)$ converges to f iff $\int_{\gamma_r} |f(z) - f_n(z)||dz| \to 0$ as $n \to \infty$ for each $r, 0 < r < 1$.

Proof. Suppose that $f_n \to f$ in $H(D)$. Since γ_r is a compact subset of D, $f_n \to f$ uniformly on γ_r. This implies that $\int_{\gamma_r} |f(z)-f_n(z)||dz| \to 0$ as $n \to \infty$ for each $r, 0 < r < 1$. Conversely, suppose that $\int_{\gamma_r} |f(z) - f_n(z)||dz| \to 0$ as $n \to \infty$ for each $r, 0 < r < 1$. Let $K \subset D$ be compact. So, there exists $r < 1$ such that $K \subset B(0;r)$. Let $r < R < 1$. Then

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(w)}{w-z}dw, \quad f_n(z) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f_n(w)}{w-z}dw$$

for all $z \in B(0;R)$ and in particular, for all $z \in K$. So, for all $z \in K$,

$$|f_n(z) - f(z)| = \frac{1}{2\pi}\left|\int_{\gamma_R} \frac{f_n(w)-f(w)}{w-z}dw\right| \leq \frac{1}{2\pi}\int_{\gamma_R} \frac{|f_n(w)-f(w)|}{|w-z|}d|w|$$

Note that since $K \subset B(0;r)$, we have $|w - z| > R - r$ for all $w \in \gamma_R$ and all $z \in K$. This implies that, for all $z \in K$,

$$|f_n(z) - f(z)| \leq \frac{1}{2\pi(R-r)}\int_{\gamma_R} |f_n(w) - f(w)|d|w|$$

But the RHS above converges to 0 as $n \to \infty$. So $f_n \to f$ uniformly on K. Since K was an arbitrary compact subset of D, this proves that $f_n \to f$ in $H(D)$. □

Problem 4. Let $\{f_n\} \subset H(G)$ be a sequence of one-one functions which converge to f. If G is a region, show that either f is one-one or f is a constant function.

Proof. Suppose that f is not a constant function. Let $z_0, z_1 \in G$ be such that $f(z_0) = f(z_1)$. Let $g_n = f_n - f(z_0)$ and $g = f - f(z_0)$. Then $g_n \to g$ in $H(G)$. $g \not\equiv 0$. Note that $g(z_0) = g(z_1) = 0$. Choose r small enough such that $\overline{B}(z_0;r) \cap \overline{B}(z_1;r) = \emptyset$ and $B(z_0;r) \cup B(z_1;r) \subset G$, and such that g has no zeroes on $|z - z_0| = r$ and $|z - z_1| = r$. Then, by Hurwitz's theorem, there exists $N >> 0$ such that g and g_N have the same number of zeroes in $B(z_0;r)$ and $B(z_1;r)$ each. So, g_N has at least one zero in $B(z_0;r)$ and $B(z_1;r)$ each. This implies that g_N cannot be one-one, a contradiction. Hence g is one-one, if it is not $\equiv 0$. Hence f is either one-one or a constant function. □

Problem 5. Suppose that $\{f_n\}$ is a sequence in $H(G), f$ is a non-constant function, and $f_n \to f$ in $H(G)$. Let $a \in G$ and $\alpha = f(a)$; show that there is a sequence $\{a_n\}$ in G such that: (i) $a = \lim a_n$; (ii) $f_n(a_n) = \alpha$ for sufficiently large n.

Proof. WLOG we can assume that $\alpha = 0$. Then $f(a) = 0$. Choose $R > 0$ such that $\overline{B}(a; R) \subset G$ and $f(z) \neq 0$ for $|z - a| = R$. By Hurwitz's theorem, choose $N_1 \in \mathbb{N}$ such that f_n has at least one zero in $B(a; R)$ for all $n \geq N_1$. Choose any one of these zeroes and denote it by a_{N_1}. Similarly, for $i \geq 1$, choose $N_{i+1} > N_i$ such that f_n has at least one zero in $B(a; R/(i+1))$ for all $n \geq N_{i+1}$. And as we did earlier, pick any zero and call it $a_{N_{i+1}}$. Now, for $n \geq N_1$, define $a_n = a_{N_i}$ for $N_i \leq i < N_{i+1}$. Then $f_n(a_n) = 0$ for all $n \geq N_1$ and $a_n \to a$. □

Problem 6. Let f be analytic on $G = \{z : \text{Re } z > 0\}$, one-one, with $\text{Re } f(z) > 0$ for all z in G, and $f(a) = a$ for some real number a. Show that $|f'(a)| \leq 1$.

Proof. By the Riemann mapping theorem, as G is simply connected, there exists an analytic bijective function $g : G \to B(0; 1)$ such that $g(a) = 0$ and $g'(a) > 0$. So, $g^{-1} : B(0; 1) \to G$ is analytic. Let $h = g \circ f \circ g^{-1} : B(0; 1) \to B(0; 1)$. Also, $h(0) = g(f(g^{-1}(0))) = g(f(a)) = g(a) = 0$. So, by Schwarz's lemma, $|h'(0)| \leq 1$. But

$$h'(0) = g'(f(g^{-1}(0)))f'(g^{-1}(0))(g^{-1})'(0) = \frac{g'(a)f'(a)}{g'(g^{-1}(0))} = f'(a)$$

So, we have $|f'(a)| \leq 1$. □

Problem 7. Let r_1, r_2, R_1, R_2 be positive numbers such that $R_1/r_1 = R_2/r_2$; show that $\text{ann}(0; r_1, R_1)$ and $\text{ann}(0; r_2, R_2)$ are conformally equivalent.

Proof. Let $G_1 = \text{ann}(0; r_1, R_1)$ and $G_2 = \text{ann}(0; r_2, R_2)$. Define $f : G_1 \to G_2$ by $f(z) = \frac{r_2}{r_1} z$. Note that f is well-defined, because if $r_1 < |z| < R_1$, then $r_2 < |\frac{r_2}{r_1} z| < \frac{R_1 r_2}{r_1} = R_2$. Clearly f is bijective and analytic, hence G_1 is conformally equivalent to G_2. □