Solutions: Homework 5

February 24, 2020

Problem 1. (a) Let G be a region, let $a \in G$ and suppose that $f : (G \setminus \{a\}) \to \mathbb{C}$ is an analytic function such that $f(G \setminus \{a\}) = \Omega$ is bounded. Show that f has a removable singularity at $z = a$. If f is one-one, show that $f(a) \in \partial\Omega$.
(b) Show that there is no one-one analytic function which maps $G = \{z : 0 < |z| < 1\}$ onto an annulus $\Omega = \{z : r < |z| < R\}$ where $r > 0$.

Proof. (a) a is an isolated singularity of f. Since Ω is bounded, $\lim_{z\to a} |f(z)| \neq \infty$, so a is not a pole. Since Ω is bounded, $\overline{\Omega} \neq \mathbb{C}$ and for any $\delta > 0$ such that $B(a;\delta) \subset G$, $\overline{f(\text{ann }(a;0,\delta))} \subset \overline{\Omega} \neq \mathbb{C}$ and so by Casorati-Weierstrass, f cannot have an essential singularity at a. Hence, f has a removable singularity at $z = a$. Now suppose that f is one-one. Suppose that $f(a) \notin \partial\Omega$. Then $f(a) \in \Omega$. So, there exists $b \neq a$, $b \in G$ such that $f(b) = f(a)$. Let $\delta > 0$ be such that $B(a;\delta) \cap B(b;\delta) = \emptyset$. Since f is one-one on $G \setminus \{a\}$, we should have $f(B(a;\delta)) \cap f(B(b;\delta)) = \{f(a)\}$. By the open mapping theorem, $f(B(a;\delta))$ and $f(B(b;\delta))$ are open in $\mathbb{C}$, and hence so should $f(B(a;\delta)) \cap f(B(b;\delta)) = \{f(a)\}$ be, which gives a contradiction. Hence $f(a) \in \partial\Omega$.
(b) Suppose such a function exists. Call it f. Then, by (a), we can extend it to $B(0;1)$ with $f(0) \in \partial\Omega$. So, $|f(0)| = r$ or R. In either case, $f(B(0;1)) = \Omega \cup \{f(0)\}$ is not open, which contradicts the open mapping theorem. So such a function cannot exist. $\square$

Problem 2. Find an analytic function f which maps $\{z : |z| < 1, \text{Re } z > 0\}$ onto $B(0;1)$ in a one-one fashion.

Proof. Let G denote the set $\{z : |z| < 1, \text{Re } z > 0\}$. Note that $G = \{z : |z| < 1\} \cap \{z : \text{Re } z > 0\}$. Note that, looking in $\mathbb{C}_\infty$, $\{z : |z| < 1\}$ is one side of the circle $|z| = 1$ while $\{z : \text{Re } z > 0\}$ is one side of the circle $\{z : \text{Re } z = 0\}$. (The "side" here refers to the right and left side that we see in the orientation principle. Since we are not really interested in what side is left and what side is right, we do not specify that. From here on, the word "side" shall be replaced with "interior".) We want to map G bijectively onto $B(0;1)$. Note that we cannot do this with a Möbius transformation as that would imply that $B(0;1)$ is the intersection of the interior of two circles in $\mathbb{C}_\infty$, which is not true, unless the two circles intersect in less than two points (Just an intuitive reasoning!). So, we hope to first use a Möbius transformation to map G onto a region that is the intersection of the interior (in $\mathbb{C}_\infty$) of two circles, that could easily be shown to be conformally equivalent to $B(0;1)$. We look for a Möbius transformation that sends the unit circle to the real axis. It is given by

$$f(z) = \frac{z+i}{1+iz}$$

Note that $f(0) = i$, so f maps $B(0;1)$ onto the upper half plane. Note that $f(0) = i, f(i) = \infty$ and $f(-i) = 0$. So f maps the imaginary axis to itself. $f(1) = 1$. So, f maps the right half plane to the right half plane. So, f maps G bijectively onto $\{z : \text{Re } z > 0, \text{Im } z > 0\}$. Now we note that the function $g : \{z : \text{Re } z > 0, \text{Im } z > 0\} \to \{z : \text{Im } z > 0\}$ given by $g(z) = z^2$ is a bijection. Now, we have a Möbius transformation that sends the upper half plane to $B(0;1)$ given by

$$h(z) = \frac{z-i}{z+i}$$

Then $F = h \circ g \circ f : G \to B(0;1)$ is analytic and bijective and $F(z) = \frac{i(z^2+2z-1)}{z^2-2z-1}$. $\square$

Problem 3. Let G_1 and G_2 be simply connected regions neither of which is the whole plane. Let f be a one-one analytic mapping of G_1 onto G_2. Let $a \in G_1$ and put $\alpha = f(a)$. Prove that for any one-one analytic map h of G_1 into G_2 with $h(a) = \alpha$ it follows that $|h'(a)| \leq |f'(a)|$. Suppose h is not assumed to be one-one; what can be said?

Proof. $g = h \circ f^{-1} : G_2 \to G_2$ with $g(\alpha) = \alpha$. Since G is simply connected, by the Riemann Mapping theorem, there exists an analytic bijective function $F : G_2 \to B(0;1)$ such that $F(\alpha) = 0$ and $F'(\alpha) > 0$. Let $g_0 = F \circ g \circ F^{-1} : B(0;1) \to B(0;1)$. Also, $g_0(0) = 0$. So, by Schwarz's lemma, $|g_0'(0)| \leq 1$. But

$$g_0'(0) = F'(g(F^{-1}(0)))g'(F^{-1}(0))(F^{-1})'(0) = \frac{F'(\alpha)g'(\alpha)}{F'(F^{-1}(0))} = g'(\alpha)$$

So, $|g'(\alpha)| \leq 1$. But, $g'(\alpha) = h'(f^{-1}(\alpha))(f^{-1})'(\alpha) = h'(a)/f'(f^{-1}(\alpha)) = h'(a)/f'(a)$. Hence,

$$|h'(a)| \leq |f'(a)|.$$

Note that in our argument above, we never had to use the fact that h is one-one. So that assumption is not necessary to conclude that $|h'(a)| \leq |f'(a)|$. $\square$