Solutions: Homework 6

February 24, 2020

Problem 1. Let f and g be analytic functions on a region G and show that there are analytic functions f_1, g_1, and h on G such that $f(z) = h(z)f_1(z)$ and $g(z) = h(z)g_1(z)$ for all z in G; and f_1 and g_1 have no common zeros.

Proof. Suppose that $f = g \equiv 0$. Then take $h \equiv 0$ and $f_1 = g_1 \equiv 1$. Now, assume that not both f and g are $\equiv 0$. Let $\{a_j\} = Z(f) \cap Z(g)$. Then $\{a_j\}$ has no limit points in G. Let $m_{f,j}$ and $m_{g,j}$ denote the multiplicities of the zeros at a_j for f and g respectively. Let $m_j = \min\{m_{f,j}, m_{g,j}\}$. By Theorem 5.15, there is an analytic function h defined on G whose only zeros are at the point a_j and a_j is a zero of h of multiplicity m_j. Let $f_1 = f/h$ and $g_1 = g/h$. Then f_1 and g_1 are analytic on $G \setminus \{a_j\}$. Note that since a_j's are zeros of f and g with multiplicity $\geq$ that of h, f_1 and g_1 have removable singularities at all a_j's. So, we define f_1 and g_1 on G and they are analytic on G. Now, we know that f_1 and g_1 have no zeros outside $\{a_j\}$. For $z = a_j$, suppose that $f_1(a_j) = g_1(a_j) = 0$. But this implies that $m_j - m_{f,j} > 0$ and $m_j - m_{g,j} > 0$, which is not possible by the definition of m_j. So, none of the a_j's can be a common zero of both f_1 and g_1. So, f_1 and g_1 have no common zeros and $f = hf_1$ and $g = hg_1$. □

Problem 2. (a) Let $0 < |a| < 1$ and $|z| \leq r < 1$; show that

$$\left|\frac{a + |a|z}{(1 - \overline{a}z)a}\right| \leq \frac{1 + r}{1 - r}$$

(b) Let $\{a_n\}$ be a sequence of complex numbers with $0 < |a_n| < 1$ and $\sum(1 - |a_n|) < \infty$. Show that the infinite product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \overline{a_n}z}\right)$$

converges in $H(B(0;1))$ and that $|B(z)| \leq 1$. What are the zeros of B?
(c) Find a sequence $\{a_n\}$ in $B(0;1)$ such that $\sum(1 - |a_n|) < \infty$ and every number $e^{i\theta}$ is a limit point of $\{a_n\}$.

Proof. (a)

$$\left|\frac{a + |a|z}{a}\right| \leq 1 + |z| \leq 1 + r \quad \text{and} \quad |1 - \overline{a}z| \geq 1 - |\overline{a}||z| \geq 1 - r$$

Combining the two, we get the inequality.
(b) Let

$$B_n = \prod_{k=1}^{n} \frac{|a_k|}{a_k}\left(\frac{a_n - z}{1 - \overline{a_n} z}\right)$$

Then $B_n \in H(B(0;1))$ and being a finite product of Möbius transformations and complex numbers of absolute value 1, we have $|B_n(z)| < 1$ for all $z \in B(0;1)$. For $n \geq 1$.

$$|B_n(z) - B_{n-1}(z)| = |B_{n-1}(z)|.\left|\frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \overline{a_n} z}\right) - 1\right| < \left|\frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \overline{a_n} z}\right) - 1\right|$$

$$= \left|\frac{a_n(|a_n| - 1) + |a_n| z(|a_n| - 1)}{a_n(1 - \overline{a_n} z)}\right| = (1 - |a_n|)\left|\frac{a_n + |a_n| z}{(1 - \overline{a_n} z) a_n}\right|$$

Let $K \subset B(0;1)$. Then there exists $r < 1$ such that $K \subset \overline{B}(0;r)$. Then, for $z \in K$, we have $|z| \leq r$, and hence by part (a), we have

$$|B_n(z) - B_{n-1}(z)| < \left(\frac{1+r}{1-r}\right)(1 - |a_n|)$$

for all $z \in K$. Let $\epsilon > 0$. Since $\sum(1 - |a_n|) < \infty$, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$\sum_{k=m+1}^{n} (1 - |a_k|) < \epsilon\left(\frac{1-r}{1+r}\right)$$

So, for all $z \in K$ and for $n > m \geq N$, we have

$$|B_n(z) - B_m(z)| \leq \sum_{k=m+1}^{n} |B_k(z) - B_{k-1}(z)| < \left(\frac{1+r}{1-r}\right) \sum_{k=m+1}^{n} (1 - |a_n|) < \epsilon$$

This implies that $B_n \to B$ uniformly in K. Since K is an arbitrary compact subset of $B(0;1)$, we have $B_n \to B$ in $H(B(0;1))$. Now, note that $|B_n(z)| < 1$ for all $z \in B(0;1)$. Hence $|B(z)| \leq 1$ for all $z \in B(0;1)$. Now, we know that for $n \geq N, B_n(a_N) = 0$. Hence $B(a_n) = 0$ for all $n \geq 1$. Note that $B(0) = \prod |a_n| \neq 0$ by Proposition 5.2 and Proposition 5.4. So, $B \not\equiv 0$. So, B has only countably many zeros in $B(0;1)$. Now suppose $B(a) = 0$ for some $a \notin \{a_n\}$. Then there exists $1 > R > |a|$ such that B has no zeros on $|z| = R$. Then, by Hurwitz's theorem, there exists $m \in \mathbb{N}$ such that B and B_m have the same number of zeros in $B(0;R)$. This implies that a cannot be a zero of B. So, the zeros of B are $\{a_n\}$.
Note: An equivalent way of doing this problem would be by a direct application of Theorem 5.9.
(c) Let $\{r_1, r_2, ...\}$ be an enumeration of the rationals in $[0,1)$. We define a sequence in $B(0;1)$ as follows:

$$a_n = \left(1 - \frac{1}{2^n}\right) e^{2\pi i r_n}$$

Then

$$\sum_{n=1}^{\infty} (1 - |a_n|) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$$

Let $0 \leq \theta < 2\pi$. Let $\epsilon > 0$. Then, for $n > 1 - \log_2(\epsilon)$, we have

$$\frac{1}{2^n} < \frac{\epsilon}{2}$$

Since $\{2\pi r_1, 2\pi r_2, ...\}$ is dense in $[0, 2\pi)$, the interval $(\theta - \frac{\epsilon}{2}, \theta + \frac{\epsilon}{2})$ contains infinitely many elements from $\{2\pi r_1, 2\pi r_2, ...\}$. Let $N > 1 - \log_2(\epsilon)$ be such that $2\pi r_N \in (\theta - \frac{\epsilon}{2}, \theta + \frac{\epsilon}{2})$. Then we have

$$|e^{i\theta} - a_N| \leq |e^{i\theta} - e^{2\pi i r_N}| + |e^{2\pi i r_N} - a_N| = |e^{i(\theta - 2\pi r_N)} - 1| + \frac{1}{2^N}$$

Now we use the fact that $|e^{ix} - 1| \leq |x|$ for all $x \in \mathbb{R}$. So, we have

$$|e^{i\theta} - a_N| \leq |\theta - 2\pi r_N| + \frac{1}{2^N} < \epsilon$$

This proves that every number $e^{i\theta}$ is a limit point of $\{a_n\}$. □

Problem 3. Show that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

Proof. Let

$$a_n = \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{\prod_{k=2}^{n}(k-1) \prod_{k=2}^{n}(k+1)}{\prod_{k=2}^{n} k^2}$$

Note that the first product in the numerator is $(n-1)!$ while the second one is $3.4....(n+1) = (n+1)!/2$. The product in the denominator is $(n!)^2$. Hence

$$a_n = \frac{(n-1)!(n+1)!}{2.n!.n!} = \frac{n+1}{2n}$$

Hence

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2}$$

□