HW8 Problems

1. Let \( f = \frac{1}{(z-1)(z-5)} \).

   (a). Prove that there is a sequence of rational functions \( R_n(z) \) whose poles can only occur at 2 and 6 such that
   \[
   \lim_{n \to \infty} \sup_{3 \leq |z| \leq 4} |f(z) - R_n(z)| = 0. \tag{1}
   \]

   (b). Does there exist a sequence of rational functions \( R_n(z) \) whose poles can only occur at 6 such that (1) holds? Justify your answer.

2. Let \( G = \{ z \in \mathbb{C} : |z| < 1 \text{ and } |z - \frac{1}{3}| > \frac{2}{3} \} \); and \( K \) be the closure of \( G : K = \{ z \in \mathbb{C} : |z| \leq 1 \text{ and } |z - \frac{1}{3}| \geq \frac{2}{3} \} \). Let \( A(K) \) be the space of continuous functions on \( K \) that are analytic on \( G \) equipped with the uniform norm on \( K \). For the purposes of this problem, a Laurent polynomial is a function of the form \( \sum_{n=-N}^{N} a_n z^n \). Determine whether the following are true or false. Justify your answer.

   (a). The set of polynomials is dense in \( H(G) \).

   (b). The set of polynomials is dense in \( A(K) \).

   (c). If \( f \) is analytic on a neighborhood of \( K \), then \( f \) can be uniformly approximated on \( K \) by Laurent polynomials.