The metric on \( C(G, \mathbb{C}) \)

**Defn.** If \( G \) is an open set in \( \mathbb{C} \) and \( \mathbb{C} \) is equipped with standard metric \( d(z, w) = |z - w| \).

we denote

\[
C(G, \mathbb{C}) = \left\{ \text{all continuous function } f : G \to \mathbb{C} \right\}
\]

Q: How to define a metric on \( C(G, \mathbb{C}) \)?

Recall: A metric \( d \) on a set \( X \): \( \forall x, y, z \)

1. \( d(x, y) \geq 0 \)
2. \( d(x, y) = 0 \iff x = y \)
3. \( d(x, y) = d(y, x) \)
4. \( d(x, z) \leq d(x, y) + d(y, z) \).

Naive idea:

define

\[
d(f, g) = \sup \left\{ |f(z) - g(z)| : z \in G \right\}
\]

for \( f, g \in C(G, \mathbb{C}) \).

But \( d(f, g) \) may be \( \infty \) in this way

A better way:

**Defn.** exhaustion sequence of \( G \)

A sequence of compact subset \( \{ K_n \}_{n=1}^{\infty} \) of \( G \)

is called an exhaustion sequence of compact set for \( G \)
is called an exhaustion sequence of compact set for $G$ if

(1) $K_1 \subset K_2 \subset K_3 \subset K_4 \subset \cdots$

(2) Every compact subset $K$ of $G$ is contained in some $K_n$. In particular,$$igcup_{n=1}^{\infty} K_n = G$$

Existence of Exhaustion sequence

Proposition 1.2: Let $G \subset \mathbb{C}$ be open. Set

$$K_n = \{ z \in G : d(z, \mathbb{C} - G) \geq \frac{1}{n} \}$$

Then (1) (2) holds.

Moreover,

(3) Every component of $\mathbb{C} - K_n$ contains a component of $\mathbb{C} - G$.

\[ \textbf{Pf:} \] P143 in the book

Now for each $K_n$, we can define a metric on $C(K_n, \mathbb{C})$:

$$P_n(f, g) = \sup \left\{ d(f(z), g(z)) : z \in K_n \right\}$$

for $f, g \in C(K_n, \mathbb{C})$.

Q: How to use this to define a metric on $C(G, \mathbb{C})$.

Observation: $d_n(f, g) = \frac{P_n(f, g)}{1 + P_n(f, g)}$ is also a metric on $C(K_n, \mathbb{C})$. 
Pf:  
1. \( d_n(f, g) \geq 0 \)
2. \( d_n(f, g) = 0 \iff p_n(f, g) = 0 \iff f = g \)
3. \( d_n(f, g) = d_n(g, f) \)
4. **Triangle inequality.**
   \[
   \frac{p_n(f, g)}{1 + p_n(f, g)} \leq \frac{p_n(f, h)}{1 + p_n(f, h)} + \frac{p_n(h, g)}{1 + p_n(h, g)}
   \]

We write \( a = p_n(f, g), \ b = p_n(f, h), \ c = p_n(h, g) \).

Since \( p_n \) is a metric \( \Rightarrow a \leq b + c \).

Now set \( \psi(x) = \frac{x}{1+x} \), we need to show

\[
\psi(a) \leq \psi(b) + \psi(c).
\]

Note \( \psi(x) = 1 - \frac{1}{1+x}, \ \psi'(x) = \frac{1}{(1+x)^2}, \ \psi''(x) = -2 \frac{1}{(1+x)^3} \)

\( \Rightarrow \psi \uparrow \psi' \) on \( x \in (0, +\infty) \).

Then \( \psi(a) \leq \psi(b+c) \).

We fix \( c > 0 \) and set \( \psi(b) = \psi(b) + \psi(c) - \psi(b+c) \).

Note \( \psi(0) = 0 \)

\( \psi'(b) = \psi'(b) - \psi'(b+c) \geq 0 \) as \( \psi' \uparrow \)

\( \Rightarrow \psi(b) \geq 0 \) for all \( b \geq 0 \). Thus

\[
\psi(b) + \psi(c) \geq \psi(b+c)
\]

\( \Rightarrow \psi(a) \leq \psi(b) + \psi(c) \).
Metric on $C(G, \mathbb{C})$:

Define for $f, g \in C(G, \mathbb{C})$

$$p_n(f, g) = \sup \left\{ d(f(z), g(z)) : z \in K_n \right\}$$

$$p(f, g) = \sum_{n=1}^{\infty} \frac{p_n(f, g)}{(n+1)^2}$$

Then

**Proposition 1.6:** $(C(G, \mathbb{C}), p)$ is a metric space.

**pf:** Page 144 in the book

**Proposition 1.10:**

A sequence $f_n \to f$ in $(C(G, \mathbb{C}), p)$

(i.e. $p(f_n, f) \to 0$) $\iff$

$f_n$ converges to $f$ on every compact subset of $G$.

**Idea:** When $p(f_j, f) = \sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n \frac{p_n(f_j, f)}{1 + p_n(f_j, f)} \to 0$

On every $K_n$, $\frac{p_n(f_j, f)}{1 + p_n(f_j, f)} \to 0 \Rightarrow p_n(f_j, f) \to 0$

**Proposition 1.12:** $C(G, \mathbb{C})$ is a complete metric space under $p$.

Recall: “complete” means every Cauchy sequence $\{f_j\}$ has a limit $f$ in $C(G, \mathbb{C})$.

**Idea of pf:** If $\{f_j\}$ is Cauchy in $(C(G, \mathbb{C}), p)$, then

$$p_n(f_j, f_k) \to 0 \text{ as } j, k \to \infty$$

That is, $\{f_j\}$ is Cauchy on every $K_n$
That is, \( \{f_j\} \) is Cauchy on every \( K_n \).

Thus \( \{f_j\} \) has a pointwise limit \( f \) on \( K_n \).

Moreover, \( f_j \to f \) on \( K \) uniformly.

Since the uniform limit of continuous functions is continuous. Thus \( f \in C(G, \mathbb{C}) \).

**Defn.** A set \( F \subseteq C(G, \mathbb{C}) \) is normal if each sequence in \( F \) has a subsequence which converges to a function \( f \) in \( C(G, \mathbb{C}) \).

**Proposition 1.15.** A set \( F \subseteq C(G, \mathbb{C}) \) is normal if and only if its closure is compact.

**Proposition 1.16.** A set \( F \subseteq C(G, \mathbb{C}) \) is normal if and only if for every compact set \( K \subseteq G \) and \( \delta > 0 \) there are functions \( f_1, \ldots, f_n \) in \( F \) such that for \( f \in F \), there \( \exists \) some \( 1 \leq k \leq n \) st

\[
\sup \{d(f_1(z), f_k(z)) : z \in K\} < \delta.
\]

Idea of Pf: Use the compactness of \( \overline{F} \).

**Defn.** A set \( F \subseteq C(G, \mathbb{C}) \) is equicontinuous at a point \( z_0 \in G \) if for every \( \varepsilon > 0 \), there \( \exists \delta > 0 \) s.t. for \( |z - z_0| < \delta \),

\[
d(f(z), f(z_0)) < \varepsilon.
\]
\[ d(f(z), f(z_0)) < \varepsilon \]
for every \( f \in F \).

2. \( F \) is equicontinuous over a set \( E \subseteq \mathbb{C} \) if
for every \( \varepsilon > 0 \) there \( \exists \delta > 0 \) s.t.
for \( z \) and \( z' \) in \( E \) and \( |z - z'| < \delta \),
\[ d(f(z), f(z')) < \varepsilon \]
for all \( f \in F \).

**Proposition 1.22.** Suppose \( F \subseteq C(G, \mathbb{C}) \) is equicontinuous at
every pt of \( G \) then \( F \) is equicontinuous over
every compact subset of \( G \).

**Arzela-Ascoli Thm.** A set \( F \subseteq C(G, \mathbb{C}) \) is normal iff
the following two conditions are satisfied:
(a) for each \( z \in G \), \( \exists M > 0 \) s.t.
\[ |f(z)| \leq M \] for all \( f \in F \).
(b) \( F \) is equicontinuous at each pt of \( G \)
\[ \iff F \] is equicontinuous over every compact subset \( K \subseteq G \).

**Def:** We will only prove the \( \subseteq \) direction. Suppose \( F \) satisfies (a) and (b). We will show \( F \) is normal.

Let \( \{ z_n \}_{n=1}^{\infty} \) be the sequence of all pts in \( G \) with
rational real and imaginary parts. Then
\( z_n \in G \) for all \( n \).
rational real and imaginary parts. Then 
\{2n\} is dense in \( \mathbb{G} \). Let \( \{f_k\} \) be a sequence in \( \mathbb{F} \).

We find a subsequence \( f^{(1)} \) of \( \{f_k\} \)

\[ f^{(1)}_j : f_{k_1}, f_{k_2}, f_{k_3}, \ldots, f_{k_j} \ldots \text{ which converges at } Z_1. \]

Then we find a subsequence \( f^{(2)}_j \) of \( \{f^{(1)}_j\} \) s.t

\[ f^{(2)}_j \text{ converges at } Z_2. \]

Inductively, we find a subsequence \( f^{(i)}_j \) of \( \{f^{(i-1)}_j\} \) s.t \( f^{(i)}_j \) converges at \( Z_i \). Then by diagonalization method, we can find a subsequence \( \{f_i\} \)

of \( \{f_k\} \) s.t

\[ f^{(1)}, f^{(2)}, \ldots, \text{ converges at every } Z_n. \]

We will still call this subsequence \( \{f_k\} \).

Next we prove

**Lemma:** Fix a compact subset \( K \subset \mathbb{G} \).

Let \( \varepsilon > 0 \). \( \exists J > 0 \) s.t

\[ k, j \geq J \Rightarrow \sup_{z \in K^j} d(z, f_j) < \varepsilon. \]

**Proof:** Write \( R = d(K, \mathbb{G} \setminus K) > 0 \)

Let \( K_1 = \{z \in \mathbb{G} : d(z, K) \leq \frac{R}{2} \} \)

Then \( K_1 \) is compact and

\[ K \subset K_0 \subset K_1 \subset \mathbb{G}. \]

By assumption (b), \( \mathbb{F} \) is equicontinuous on \( K_1 \).
By assumption (b), \( f \) is equicontinuous on \( K_1 \).

Thus \( \exists \sigma > 0 \) with \( 0 < \sigma < \frac{\varepsilon}{2} \) s.t.

\[
d(f(z_2), f(z_1)) < \frac{\varepsilon}{2}
\]

for all \( f \in F \) and \( z, z' \in K_1 \) with \( |z - z'| < \sigma \).

Let \( D = \{ z_n : z_n \in K_1 \} \).

If \( z \in K \) then \( \exists z_n \) with \( |z - z_n| < \sigma \). But \( \sigma < \frac{\varepsilon}{2} \)

\[
\Rightarrow d(z_n, K) < \frac{\varepsilon}{2}. \text{ Thus } z_n \in K_1.
\]

As \( z \) is arbitrary in \( K \), we have \( \{ B(w, \sigma) : w \in D \} \)

is an open cover of \( K \). As \( K \) is compact, \( \exists \) a

finite cover of \( K \), i.e., \( \exists w_1, \ldots, w_n \in D \) s.t

\[
K \subseteq \bigcup_{i=1}^{n} B(w_i, \sigma).
\]

Since \( \lim_{k \to \infty} f_k(w_i) \) exists for \( 1 \leq i \leq n \), there \( \exists \sigma > 0 \)

s.t \( j, k \geq \sigma \Rightarrow d(f_k(w_i), f_j(w_i)) < \frac{\varepsilon}{3} \) for all

\( 1 \leq i \leq n \).

Let \( z \) be an arbitrary pt in \( K \). Then \( \exists \) some

\( 1 \leq i \leq n \) s.t \( z \in B(w_i, \sigma) \). Then

\[
k, j \geq \sigma \Rightarrow
\]

\[
d(f_k(z_2), f_j(z_2)) \leq d(f_k(z_2), f_k(w_i)) + d(f_k(w_i), f_j(w_i)) + d(f_j(w_i), f_j(z_2))
\]

\[
< \varepsilon.
\]

This proves the lemma.

By the lemma, \( \{ f_k \} \) has a pointwise limit \( f \) s.t

\( f_k \to f \) uniformly on \( K \). By proposition 1.10,
$f_k \to f$ uniformly on $K$. By proposition 1.10,

$f_k \to f$ in $C(K, \mathbb{C})$. 