7.4 The Riemann Mapping Thm

**Defn 4.1.** A region $G_1$ is conformally equivalent to $G_2$ if there exists an analytic function $f : G_1 \to \mathbb{C}$ such that $f$ is one-to-one and $f(G_1) = G_2$.

**Remark:** Conformal equivalence is an equivalence relation.

**Ex.** Let $G_1 = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$

$G_2 = \{z \in \mathbb{C} : \text{Re} z > 0\}$

Let $f(z) = \sqrt{z}$ the principal branch of the square root.

**Thm 4.2 (Riemann Mapping Thm)** Let $G$ be a simply connected region with $G \neq \mathbb{C}$. Let $a \in G$. Then there exists a unique analytic function $f : G \to \mathbb{C}$ having the properties:

(a) $f(a) = 0$ and $f'(a) > 0$

(b) $f$ is 1-1

(c) $f(G) = \{z : |z| < 1\}$

**Proof:** The uniqueness part is easy. If $f, g$ both satisfy (a), (b), (c), then $f \circ g^{-1} : D \to D$ is analytic, 1-1, onto.
Also \( f \circ g^{-1}(0) = 0 \).

By Schwarz's lemma, \( f \circ g^{-1} = C \), where \( |C| = 1 \).

\[ \Rightarrow f = C \circ g(z). \]

Differentiate at \( z = a \); \( \Rightarrow \)

\[ f'(a) = C \circ g'(a). \]

Note \( f'(a) > 0, g'(a) > 0 \Rightarrow C = 1. \)

Thus \( f = g \).

Step 1: we first prove \( \exists \varphi \in H(G), 1-1 \) on \( G \), s.t

\[ \varphi(G) \text{ is bdd}. \]

Note as \( G \neq \mathbb{C}, \exists b \in C \text{ s.t } b \notin G. \text{ Thus } h \equiv z - b \text{ never vanishes in } G. \text{ Recall on P84, Corollary 6.17.} \)

If \( G \) is simply connected, \( h \in H(G) \) and \( h(z) \neq 0, \forall z \in G, \) then \( \exists \hat{h} \in H(G) \text{ s.t } h = e^{\hat{h}}. \) Moreover, \( \hat{h} \) can be defined by \( e^{\hat{h}} = \log w. \) Hence

\[ \sqrt{z - b} \text{ has a well-defined branch in } G. \]

Call it \( g(z) \). Note \( g \) is 1-1.

Moreover, write \( g(G) = E \) then \( 0 \notin E \).

Claim: If \( w \in E, -w \notin E \).

Proof: Suppose \( w, -w \in E, \) then \( \exists z_1, z_2, \text{ s.t } w = \sqrt{z_1 - b}, -w = \sqrt{z_2 - b} \)

\[ \Rightarrow \sqrt{z_1 - b} = \sqrt{z_2 - b} \Rightarrow z_1 - b = z_2 - b \]

\[ \Rightarrow z_1 = z_2 \Rightarrow w = -w \Rightarrow w = 0 \]
\[ \Rightarrow z_1 = z_2 \Rightarrow w = -w \Rightarrow w = 0 \]

This is a contradiction.

By open mapping thm. E is a region in C.

As \( a \in G \Rightarrow \text{gra} \) is an interior pt of E

\[ \exists \delta > 0 \text{ s.t. } B(\text{gra}) ; \delta \subset E. \]

Thus \( B(-\text{gra}) ; \delta \subset C - E \Rightarrow \text{dist}(-\text{gra}, E) \geq \delta \)

Thus when \( z \in E \), we have \( |g(z) + \text{gra}| \geq \delta \).

Let \( \Psi(z) = \frac{1}{g(z) + \text{gra}} \)

Thus if \( z \in G \)

\[ |\Psi(z)| \leq \frac{1}{\delta} \text{ Thus } \Psi(G) \text{ is bdd.} \]

Note: \( \Psi \) is 1-1, as \( g(z) \) is 1-1.

**Step 2: Assume \( G \) is bdd.**

To prove the existence,

Idea: Consider the family \( F \) of all analytic functions having properties (a), (b), and maps \( G \) to \( D \).

Then choose a special element \( f \) in \( F \) satisfies (c).

Let \( F = \{ f : f \in H(G), 1-1, \text{gra} = 0, f'(a) > 0, f(G) \subset 1D \} \).

**Claim:** \( F \neq \emptyset \).

**Pf:** Assume \( G \subset B(a; R) \) for some \( R > 0 \).

Let \( h(z) = \frac{z - a}{R} \). Then \( h \in F \).
Now fix \( r > 0 \) s.t. \( B(a; r) \subset G \). By Cauchy’s Estimate, we obtain for all \( f \in F \),

\[
\left| f'(a) \right| \leq \frac{1}{r} \sup \left\{ |f(z)| : z \in B(a; r) \right\} \leq \frac{1}{r}.
\]

Thus \( \{f'(a) : f \in F\} \) has an upper bound. \( \Rightarrow \)

\[
M = \sup \{f'(a) : f \in F\} < \infty. \text{ Note } M > 0.
\]

Next we will prove \( f_* \in F \) such that \( f'_*(a) = M \). Indeed, \( \exists \{f_n\} \subset F \) such that \( \lim_{n \to \infty} f'_n(a) = M \). But \( |f(z)| \leq 1 \) on \( G \) for all \( f \in F \). By Montel’s thm. we have

\( \{f_n\} \) has a convergent subsequence in \( H(G) \). We still call it \( \{f_n\} \). Write the limit as \( f_* \):

\[
f_n \to f_* \text{ in } H(G).
\]

We also have \( f'_n \to f'_* \text{ in } H(G) \). In particular,

\[
f'_*(a) = \lim_{n \to \infty} f'_n(a) = M > 0.
\]

As each \( f_n \) is 1-1, \( f_* \) constant, by what we proved last time,

\( f_* \) is also 1-1.

Moreover, \( f_*' (a) = \lim_{n \to \infty} f_n'(a) = 0 \).

**Claim:** \( |f'_*(z)| < 1 \) for all \( z \in G \).

**Proof:** As \( |f_n'(z)| < 1 \) for any \( z \in G \), \( n \geq 1 \)
Proof: As $|f'(z)| < 1$ for any $z \in G, n \geq 1$

Let $n \to \infty \Rightarrow |f(z)| \leq 1$ for $z \in G$.

Note $f$ is not constant. By M.M.T

$|f(z)| < 1$ for $z \in G$.

It remains to prove $f(G) = D$.

**Step 3:** It remains to show that $f(G) = D$. Suppose we D.S.t $w \neq f(G)$. Then the function

$$h(z) = \frac{f(z) - w}{1 - \overline{w}f(z)}$$

is analytic in $G$ and never vanishes. Then $\exists$ an analytic function $h: G \to \mathbb{C}$ such that

$$h^2 = \frac{f(z) - w}{1 - \overline{w}f(z)} \quad (\star)$$

Note $h^2$ maps $G$ to $D$, is $1-1 \Rightarrow h$ maps $G$ to $D$, is $1-1$.

Define $g: G \to \mathbb{C}$ by

$$g(z) = \frac{h'(a)h(z) - h(a)}{h'(a)(1 - h(a)h(z))}$$

Then $g(G) \subset D$, $g(a) = 0$, and is $1-1$.

Also

$$g'(a) = \frac{h'(a)}{h'(a)} \frac{h'(a) [1 - h(a)^2]}{[1 - h(a)^2]^2} = \frac{h'(a)}{1 - h(a)^2}$$

But $f(a) = 0 \Rightarrow |h(a)|^2 = |w|$. 

But \( f(a) = 0 \Rightarrow |h(a)|^2 = |w| \).

Differentiate \((*)\) \( \Rightarrow \)

\[
2 h(a) h'(a) = f'(a) (1-|w|^2)
\]

\( \Rightarrow \)

\[
|h'(a)| = \frac{f'(a) (1-|w|^2)}{2 \sqrt{|w|}}
\]

\( \Rightarrow \)

\[
g'(a) = \frac{f'(a) (1-|w|^2)}{2 \sqrt{|w|}} \frac{1}{1-|w|}
\]

\[
= f'(a) \frac{1+|w|}{2 \sqrt{|w|}}
\]

\( \Rightarrow f'(a) \).

Note \( g \notin F \) and this contradicts with the choice of \( f \).

Hence it must hold that \( f(G) = \{1\} \).