Practice Problems for Midterm

1. (i). State the Residue Theorem.

(ii). Compute \( \int_0^\infty \frac{\sin x}{x} \)

(iii). Compute the residue of \( f(x) = \frac{x^2}{(x^2+1)^2} \) at \( z = -i \).

2.(i) State Schwarz's Lemma.

(ii). Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( f \in H(\mathbb{D}) \). Assume \( f(\frac{1}{2}) = 0 \) and \( |f(z)| \leq 1 \) in \( \mathbb{D} \). Prove \( |f(\frac{1}{4})| \leq \frac{2}{7} \).

3. (i). State any version of Maximum Modulus Theorem.

(ii) Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( f \) be analytic in \( \mathbb{D} \). Assume \( f(0) = 1 \) and \( \text{Re } f \geq 0 \) on \( G \). Prove \( \text{Re } f > 0 \).

(iii). Let \( f \) be as in part (ii). Use Schwarz lemma to prove

\[
|f(z)| \leq \frac{1 + |z|}{1 - |z|}.
\]

4. (i) State Montel’s Theorem.

(ii). Prove the following family is normal:

\[
\{ f \in H(\mathbb{D}) : f(0) = 1, \text{Re} f \geq 0 \}
\]

5.(i). State Hurwitz’s Theorem.

(ii) Let \( f_n \to f \) in \( H(G) \). Assume \( f_n \) is one-to-one on \( G \) for all \( n \) and \( f \) is not constant. Prove \( f \) is one-to-one.

6. Page 173, HW 4 (a), (b) in the book.

7.(i) State the Riemann mapping theorem.

(ii). Find an analytic map \( f \) on the unit disk \( \mathbb{D} \) such that \( f(\mathbb{D}) = \mathbb{C} \).

8. All homework problems and especially the graded ones.