Pratice Problems

1. (i). State the Residue Theorem.
   (ii). Compute \( \int_{0}^{\infty} \frac{\sin x}{x} \)
   (iii). Compute the residue of \( f(x) = \frac{x^2}{z^2+1} \) at \( z = -i \).

2. (i) State Schwarz’s Lemma.
   (ii). Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( f \in H(D) \). Assume \( |f(z)| \leq 1 \) in \( D \). How large \( |f'(\frac{1}{2})| \) can be?

3. Prove there are infinitely many solutions in \( \mathbb{C} \) to the equation \( \sin(z) = \sin(iz) \).

4. Suppose \( f \) has an essential singularity at 0, and \( g \) has an essential singularity at 0. Prove that at least one of the functions \( f + g \) and \( fg \) has an essential singularity at \( z = 0 \).

5. Suppose \( f \) is a nonconstant entire function. Which of the following must be countably infinite?
   (a). \( f(\mathbb{Z}) \)  (b). \( f(\mathbb{Q}) \)  (c). \( f^{-1}(\mathbb{Q}) \)

6. Prove that for any \( a \in \mathbb{C} \) and any integer \( n \geq 2 \) the polynomial \( 1 + z + az^n \) has at least one root in the disk \( \{|z| \leq 2\} \). (Hint: use the Vieta theorem that says that the product of the roots of a monic polynomial is equal to its constant term in absolute value.)

7. (i). Let \( f \) be a non-constant holomorphic function on a neighborhood of the closed unit disk such that \( |f(z)| \) is constant on the unit circle. Prove that \( f \) has at least one zero in the unit disk.
   (ii). Find all entire \( f \) such that \( |f| \) is constant on the unit circle.

8. Let \( f \) be a holomorphic function in the unit disk \( \mathbb{D} \) that is injective and satisfies \( f(0) = 0 \) Prove that there exists a holomorphic function \( g \) in \( \mathbb{D} \) such that \( (g(z))^2 = f(z^2) \) for all \( z \in \mathbb{D} \).


10. All homework problems and midterm problems.