Practice Problems for final exam

1. True or False. Justify your answer.
   
   (a). Let \( G = \mathbb{C} \setminus \{ z \in \mathbb{R} : z \text{ is an integer} \} \). Suppose \( f \in H(G) \) such that \(|f(z)| \leq 1\) for all \( z \in G \). Then \( f \) is a constant.
   
   (b). Let \( G = \{ 0 < |z - 1| < 1 \} \). There exists \( f \in H(G) \) such that \( \lim_{z \to 1} |(z - 1)^k f(z)| = \infty \), for all integer \( k \geq 1 \).

2. Suppose \( f \) has an essential singularity at 0, and \( g \) has an essential singularity at 0. Prove that at least one of the functions \( f + g \) and \( fg \) has an essential singularity at \( z = 0 \).

3. Suppose \( f \) is a nonconstant entire function. Which of the following must be countably infinite?
   
   (a). \( f(\mathbb{Z}) \)  
   (b). \( f(\mathbb{Q}) \)  
   (c). \( f^{-1}(\mathbb{Q}) \)

4. Let \( f \) be a holomorphic function in the unit disk \( \mathbb{D} \) that is injective and satisfies \( f(0) = 0 \). Prove that there exists a holomorphic function \( g \) in \( \mathbb{D} \) such that \( (g(z))^2 = f(z^2) \) for all \( z \in \mathbb{D} \).

5. State Riemann Mapping Theorem, Runge’s Theorem, Weierstrass Factorization Theorem, Mittag-Leffler’s Theorem, Schwarz Reflection Principle.

6. Let \( \{ p_n \}_{n=1}^{\infty} \subset \mathbb{Z}^+ \) be the sequence of prime numbers. Prove there exists \( f \in H(\mathbb{C}) \) such that \( f(p_n) = p_{n+1} \) for each \( n \).

7. Let \( f \) be a continuous function on \( \{ 0 < |z| \leq 1 \} \) that is analytic function on \( \{ 0 < |z| < 1 \} \). Assume \( f(z) = 0 \) for every \( z = e^{i\theta} \) with \( \frac{\pi}{4} < \theta < \frac{\pi}{3} \). Prove \( f \equiv 0 \).

8. Let \( A_1 = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) and \( A_2 = \{ z \in \mathbb{C} : 1 < |z| < 2 \} \). Prove \( A_1 \) and \( A_2 \) are not conformally equivalent.

9. Describe all analytic functions on \( \mathbb{C} \setminus \{ 0 \} \) with the property that
   
   \[ |f(z)| \leq C(|z|^2 + \frac{1}{|z|^2}) \]
   
   for some constant \( C > 0 \).