Pratice Problems

Note: In this exam, \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \).

1. Compute the integral \( \int_0^\infty \frac{1}{x^3 + 1} \, dx \).

2. Show that if \( G \neq \mathbb{C} \) is a simple connected region in \( \mathbb{C} \), \( f : G \to G \) is analytic, and \( f(z) \) is not identically zero equal \( z \), then \( f \) has at most one fixed point in \( G \). Show by examples that the conclusion fails if \( G = \mathbb{C} \) or \( G \) is not simple connected.

3. Let \( f(z) \) be analytic with \( |f(z)| \leq \frac{1}{2} \) on \( \mathbb{D} \) and \( f(0) = r \in \mathbb{R} \). Here \( 0 < r < \frac{1}{2} \).
   
   (a). Prove that \( f(z) \) has no zero in the disk \( \{|z| < 2r\} \).
   
   (b). Can \( f(z) \) have a zero on the circle \( \{|z| = 2r\} \)? If so, find all such functions \( f(z) \).

4. Let \( b \in \mathbb{D} \) and set \( f(z) = z^7 - 2z^5 + b \).
   
   (a). How many simple roots \( f \) has in \( \mathbb{D} \)?
   
   (b). How many simple roots \( f \) has in \( \{1 \leq |z| < 2\} \)?

5. All homework problems.