Section 6.1

4. Let \( f(z) = \frac{1}{z(z-1)(z-2)} \); give the Laurent Expansion of \( f(z) \) in each of the following annuli:

(b) \( \text{ann}(0; 1, 2) \)

Solution: We start by finding the Laurent expansion for \( g(z) := \frac{1}{(z-1)(z-2)} \).

Shorter Method: Since \(|\frac{z}{2}| \) and \(|\frac{1}{z}| \) are both < 1 in this annulus, the usual geometric series expansions are valid, and we write

\[
\frac{1}{z-1} = \frac{1}{z} \frac{1}{1-\frac{1}{z}} = \sum_{n=1}^{\infty} z^{-n};
\]

\[
\frac{1}{z-2} = \frac{1}{2} \left( \frac{1}{\frac{z}{2}} - 1 \right)
= -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}};
\]

The partial decomposition of \( g(z) \) is

\[ g(z) = \frac{1}{z-2} - \frac{1}{z-1} \]

so the Laurent expansion is

\[
\frac{1}{z(z-1)(z-2)} = \frac{1}{z} \left( -\sum_{n=-\infty}^{-1} z^{-n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \right)
= -\sum_{n=-\infty}^{-2} z^{-n} - \sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}}.
\]

Longer Method: Another way to do this is to directly calculate the coefficients using

\[
a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{z^{n+1}} dz
\]
for each $n$, with $\gamma$ the circle of radius $\frac{3}{2}$ around the origin, say. For $n \leq -1$ this is not so bad: $g_n(z) := \frac{(z-1)g(z)}{z^{n+1}}$ is analytic in $B(0, \frac{3}{2}) \setminus \{1\}$, while $\gamma$ is path-homotopic to the circle of radius $\frac{1}{2}$ around $1$, and so by Cauchy’s formula

$$a_n = g_n(1) = \frac{1}{-1} = -1.$$  

For $n \geq 0$, we can use partial fractions to show that

$$\frac{g(z)}{z^{n+1}} = \sum_{i=1}^{n+1} \frac{2^{n+2-i} - 1}{z^{n+1} z^i} - \frac{1}{z - 1} + \frac{1}{2^{n+1} (z - 2)}.$$  

More specifically, this follows inductively from the partial fraction decompositions

$$\frac{1}{z(z - 1)(z - 2)} = \frac{1}{2z} - \frac{1}{z - 1} + \frac{1}{2(z - 2)}$$

$$\frac{1}{z(z - 1)} = \frac{1}{z - 1} - \frac{1}{z}$$

$$\frac{1}{z(z - 2)} = \frac{1}{2(z - 2)} - \frac{1}{2z}.$$  

It thus remains to integrate each summand separately. Cauchy’s formula tells us that all but two of the terms vanish, leaving

$$a_n = \frac{1}{2\pi i} \left( \int_{\gamma} \frac{2^{n+1} - 1}{2^{n+1} z} \, dz - \int_{\gamma} \frac{1}{z - 1} \, dz \right) = \frac{2^{n+1} - 1}{2^{n+1}} - 1 = \frac{1}{2^{n+1}}.$$  

Dividing by $z$, the resulting Laurent series is

$$\sum_{n=-\infty}^{\infty} a_{n+1} z^n.$$
1. Calculate the following integrals:

(a) \[ \int_0^\infty \frac{x^2}{x^4 + x^2 + 1} \, dx \]

**Solution:** We are integrating an even function, the desired integral is half the integral from \(-\infty \) to \(\infty \). The advantage to this is that we can apply the residue theorem to the usual semicircular contour. \(x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)\), the 3rd and 6th cyclotomic polynomials, so its roots are the primitive 3rd and 6th roots of unity. Two of these, \(\zeta := e^{\pi i/3}\) and \(\zeta^2\), are in the upper half plane, so letting \(\gamma_R\) be the upper semicircle, we see that

\[
\int_{-R}^R \frac{x^2}{x^4 + x^2 + 1} \, dx + \int_{\gamma_R} \frac{z^2}{z^4 + z^2 + 1} \, dz = 2\pi i (\text{Res}(\frac{z^2}{z^4 + z^2 + 1}; \zeta) + \text{Res}(\frac{z^2}{z^4 + z^2 + 1}; \zeta^2))
\]

We first bound \(\int_{\gamma_R} \frac{z^2}{z^4 + z^2 + 1} \, dz\), using the triangle inequality to find that for \(R\) sufficiently large,

\[
|\int_{\gamma_R} \frac{z^2}{z^4 + z^2 + 1} \, dz| \leq \int_{\gamma_R} \frac{R^2}{|R^4 e^{2it} + R^2 e^{2it} + 1|} \, dz
\]

\[
\leq \frac{R^2}{|R^4 - |R^2 e^{2it} + 1||} \, dz
\]

\[
\leq \frac{R^2}{R^4 - R^2 - 1}
\]

\[
= \pi \frac{R^3}{R^4 - R^2 - 1}
\]

which certainly goes to 0 as \(R \to \infty\).

Next we calculate the residues. The bottom polynomial splits completely, so each singularity is a simple pole, and therefore

\[
\text{Res}(\frac{z^2}{z^4 + z^2 + 1}; \zeta) = \lim_{z \to \zeta} \frac{(z - \zeta)z^2}{(z^2 + z + 1)(z^2 - z + 1)}
\]

\[
= \frac{\zeta^2}{(\zeta^2 + \zeta + 1)(\zeta - \zeta^{-1})}
\]

\[
= \frac{\zeta}{(\zeta + 1 + \zeta^{-1})(2i \text{Im} \zeta)}
\]

\[
= \frac{1}{2} + i \frac{\sqrt{3}}{2}
\]

\[
= \frac{1 + i \sqrt{3}}{4i \sqrt{3}}.
\]
By the same token,
\[
\text{Res}\left(\frac{z^2}{z^4 + z^2 + 1}; \zeta^2\right) = \frac{\zeta^4}{(\zeta^2 - \zeta^{-2})(\zeta^4 - \zeta^2 + 1)} = \frac{\zeta^4}{(i\sqrt{3})(-2\zeta^2)} = \frac{-1 + i\sqrt{3}}{2} = \frac{-2i\sqrt{3}}{4i\sqrt{3}}.
\]

Taking the limit,
\[
\int_0^\infty \frac{x^2}{x^4 + x^2 + 1}dx = \pi i \left( \frac{2}{4i\sqrt{3}} \right) = \frac{\pi}{2\sqrt{3}}.
\]

(b) \(\int_0^\infty \frac{\cos x - 1}{x^2}dx\)

**Solution:** Again, we can just take half of \(\int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2}dx\). The contour we’ll use, \(C\), is the same as a similar example in the book: a larger half circle \(\gamma_R\) and a smaller half circle \(\gamma_r\) around 0, along with the parts of \(x\)-axis between \(r\) and \(R\) and \(-r\) and \(-R\), respectively. To better handle the integral along \(\gamma_R\), note that \(\cos z\) is real part of \(e^{iz}\), which is bounded by 1. This means that
\[
|\int_{\gamma_R} \frac{e^{iz} - 1}{z^2}dz| \leq \frac{2}{R^2}
\]
goesto 0 as \(R\) goes to \(\infty\).

To deal with \(\gamma_r\) as \(r\) goes to 0, we can expand \(e^{iz} - 1\) as a power series and (using that the convergence is uniform) integrate term by term. Doing so, we see that
\[
\int_{\gamma_r} \frac{e^{iz} - 1}{z^2}dz = \sum_{k=1}^{\infty} \int_{\gamma_r} \frac{(iz)^k}{k!z^2}dz
\]
Every term except the first is (a scalar multiple of a) nonnegative power of \(z\), which means that they have as primitives positive powers of \(z\). As \(r\) goes to 0, these will
as well. The only thing remaining to calculate is

\[ \int_{\gamma_r} \frac{i}{z} \, dz = \int_\pi^0 i^2 r e^{it} \frac{1}{r e^{it}} \, dt \]

\[ = \pi \]

which shows that the limit as \( r \) goes to 0 is also \( \pi \). \( \frac{e^{iz} - 1}{z^2} \) has no singularities in the region enclosed by \( C \), so its integral along \( C \) is 0. Putting it all together, we have that

\[ \int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2} \, dx = -\Re \lim_{r \to 0} \int_{\gamma_r} \frac{e^{iz} - 1}{z^2} \, dz \]

\[ = -\pi \]

so the answer is \(-\frac{\pi}{2}\).

2. Verify the following equations:

(g) \( \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^z} \, dx = \frac{a}{\sin a\pi} \) if \( 0 < a < 1 \);

**Solution:** A different contour is required here, one that dodges the singularities at the odd multiples of \( \pi i \). The fact that the \(|e^z|\) behaves well when \( \text{Re} \, z \) is constant, suggests that we should just use a rectangle \( \gamma \) with corners at \( \pm R \) and \( 2\pi i \pm R \). Label these segments (in counterclockwise order starting from the x-axis) \( \gamma_i \), \( 1 \leq i \leq 4 \). We want to bound the integrals along \( \gamma_2 \) and \( \gamma_4 \). By design, this is easy to do, since

\[ | \int_{\gamma_1} \frac{e^{az}}{1 + e^z} \, dz | \leq \frac{e^{aR}}{e^R - 1} \]

\[ \leq \frac{e^{aR}}{2eR} \]

\[ = \frac{1}{2} e^{(a-1)R} \],

whose limit as \( R \) goes to infinity is indeed 0 because \( a < 1 \). Similarly, we find that

\[ | \int_{\gamma_4} \frac{e^{az}}{1 + e^z} \, dz | \leq \frac{e^{-aR}}{1 - e^{-R}} \]

which goes to 0 as \( R \to \infty \) since the numerator goes to 0 (because \( 0 < a \)) and the denominator to 1.

What of \( \gamma_3 \)? This is actually a multiple of the integral over \( \gamma_1 \), since \(-\gamma_3(t) = 2\pi i + t, -R \leq t \leq R\) gives us that
\[
\int_{\gamma_3} \frac{e^{az}}{1 + e^z} \, dz = -\int_{-R}^{R} \frac{e^{a2\pi i}e^{at}}{1 + e^{2\pi i}} \, dt
\]
\[
= -e^{a2\pi i} \int_{-R}^{R} \frac{e^{at}}{1 + e^t} \, dt
\]
\[
= -e^{a2\pi i} \int_{\gamma_1} \frac{e^{at}}{1 + e^t} \, dt.
\]

Letting \( R \) go to \( \infty \), and recalling the location of the singularities, we see that
\[
\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{1}{1- e^{a2\pi i}} 2\pi i \text{Res}(\frac{e^{az}}{1+e^z}; \pi i).
\]

To calculate the residue, we first note that \( 1 + e^z \) has a simple zero at \( z = \pi i \), since its derivative at \( \pi i \) is \(-1 \neq 0\). Thus \( \frac{e^{az}}{1+e^z} \) has a simple pole, and it follows that we can write
\[
\frac{e^{az}}{1+e^z} = \frac{e^{az}}{-(z - \pi i) + \sum_{k=2}^{\infty} a_k(z - \pi i)^k}.
\]

Proposition 2.4 then implies that the residue is just \(-e^{a\pi i}\). This means that
\[
\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} \, dx = \frac{-2\pi i e^{a\pi i}}{1- e^{a2\pi i}}
\]
\[
= \frac{-\pi 2i}{e^{-a\pi i} - e^{a\pi i}}
\]
\[
= \frac{\pi}{\sin a\pi}.
\]