Section 6.1

7. Let \( f \) be analytic in the disk \( B(0; R) \) and for \( 0 \leq r < R \) define \( A(r) = \max\{\Re f(z) : |z| = r\} \). Show that unless \( f \) is a constant, \( A(r) \) is strictly increasing.

**Solution:** Let \( g(z) = e^{f(z)} \). Then \( A(r) \) is the maximum of \( |g(z)| \) on \( \overline{B}(0; r) \) by the maximum modulus theorem, which also says that it is only attained on \( \{ |z| = r \} \). Thus \( A(r) \) is strictly increasing unless \( g \) is constant. In that case, \( f \) is constant as well (for instance, by the chain rule, or the fact that the image of \( f \) is both connected and discrete, or the fact that \( \Re f \) is constant).

Section 6.2

1. Suppose \( |f(z)| \leq 1 \) for \( |z| < 1 \) and \( f \) is a non-constant analytic function. By considering the function \( g : D \to D \) defined by

\[
g(z) = \frac{f(z) - a}{1 - \overline{a} f(z)}
\]

where \( a = f(0) \), prove that

\[
\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}
\]

for \( |z| < 1 \).

**Solution:** The definition of \( g \) does most of the work here; it allows us to apply Schwarz’s lemma and conclude that

\[
\frac{|f(z) - a|}{|1 - \overline{a} f(z)|} = |g(z)| \leq |z|
\]

for all \( z \in D \). Clearing denominators and rearranging (by assumption, both denominators are positive), the desired inequalities are equivalent to

\[
\frac{|f(0)| - |f(z)|}{1 + |f(0)||f(z)|} \leq |z| \leq \frac{|f(z)| - |f(0)|}{1 + |f(0)||f(z)|}.
\]
The first is obvious unless \(|f(0)| \geq |f(z)|\); in this case, the reverse triangle inequality says that
\[
|f(0)| - |f(z)| = |a| - |f(z)|
\leq |f(z) - a|
\]
so that
\[
\frac{|f(0)| - |f(z)|}{1 + |f(0)||f(z)|} \leq \frac{|f(z) - a|}{1 + |af(z)|}
\leq \frac{|f(z) - a|}{|1 - af(z)|}
\]
which we know to be \(\leq |z|\). Assuming instead that \(|f(z)| \geq |f(0)|\), the second inequality follows in exactly the same manner.

2. Does there exist an analytic function \(f : D \to D\) with \(f\left(\frac{1}{2}\right) = \frac{3}{4}\) and \(f'(\frac{1}{2}) = \frac{2}{3}\)?

Solution: No such \(f\) exists. If it did, we could let \(g = \varphi_{\frac{3}{4}} \circ f\). Then by Schwarz’s lemma,
\[
1 \geq |g'(\frac{1}{2})|
= |\varphi_{\frac{3}{4}}'(\frac{3}{4})f'(\frac{1}{2})|
= (1 - (\frac{3}{4})^2)^{-1} \cdot \frac{2}{3}
= \frac{16}{7} \cdot \frac{2}{3}
= \frac{32}{21}
\]
a contradiction.

8. Is there an analytic function \(f\) on \(B(0; 1)\) such that \(|f(z)| < 1\) for \(|z| < 1\), \(f(0) = \frac{1}{2}\), and \(f'(0) = \frac{3}{4}\)? If so, find such an \(f\). Is it unique?

Solution: Proposition 2.2 says that \(\varphi_{-\frac{1}{2}}\) does the job. Moreover, the argument preceding theorem 2.5 tells us that any analytic \(f : D \to D\) with \(f(0) = \frac{1}{2}\) satisfies
\[
f(0) \leq \frac{3}{4},
\]
with equality only when \(f(z) = \varphi_{-\frac{1}{2}}(cz), |c| = 1\). This forces \(f'(0) = c \cdot \frac{3}{4}\), so in fact \(c = 1\) and \(f = \varphi_{-\frac{1}{2}}\) is unique.