7. Let \( \{f_n\} \subset C(G, \Omega) \) and suppose that \( \{f_n\} \) is equicontinuous at each point of \( G \). If \( f \in C(G, \Omega) \) and \( f(z) = \lim f_n(z) \) for each \( z \) then show that \( f_n \to f \).

**Solution:** To show that \( f_n \to f \) in \( C(G, \Omega) \), it’s enough to show that \( f_n \) converges to \( f \) uniformly on compact subsets of \( G \). To see this, first note that for any \( z, z' \in G \) and any \( n \),
\[
d(f_n(z), f(z)) \leq d(f_n(z), f_n(z')) + d(f_n(z'), f(z')) + d(f(z'), f(z))
\]
by the triangle inequality. We can (uniformly) bound each of the right hand terms separately. Let \( K \subseteq G \) be any compact subset. Since \( \{f_n\} \) is equicontinuous at each point of \( G \), we know it is in fact equicontinuous on \( K \), so for any \( \epsilon \) we can find some \( \delta_1 \) independent of \( z \) such that \( z, z' \), \( d(z, z') < \delta_1 \) implies \( d(f_n(z), f_n(z')) < \epsilon \). On the other hand, \( f \) is uniformly continuous on \( K \), so we can also find a uniform \( \delta_2 \) such that \( d(f(z'), f(z)) < \epsilon \) if \( d(z, z') < \delta_2 \). Thus, given \( z \in K \), we can ensure that all three terms are bounded sufficiently close to \( z \).

However, \( K \) is compact, so we can cover \( K \) with a finite number of sufficiently small balls—of radius \( \delta \) (smaller than \( \delta_1 \) and \( \delta_2 \)) centered at \( z_1 \) through \( z_m \), say. By assumption, for each \( j \) there is an \( N_j \) such that \( n \geq N_j \) implies \( d(f_n(z_j), f(z_j)) < \epsilon \), and we let \( N \) be the greatest of these. Then the previous paragraph (letting \( z' \) be the appropriate \( z_j \)) shows that \( d(f_n(z), f(z)) < 3\epsilon \) for \( n \geq N \) uniformly on \( K \). This argument works (possibly using different \( \delta \)'s, \( N \)'s, and \( m \)'s) for any \( \epsilon \) and for any \( K \), so we have uniform convergence on compact subsets.

8. (a) Let \( f \) be analytic on \( B(0; R) \) and let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( |z| < R \). If \( f_n(z) = \sum_{k=0}^{n} a_k z^k \), show that \( f_n \to f \) in \( C(G, \mathbb{C}) \).

**Solution:** Again, this is the same as asking for uniform convergence of the partial sums on compact subsets of \( B(0; R) \). Any such subset is certainly contained in \( \overline{B}(0; r) \) for some \( r < R \), and uniform convergence on these follows from the Weierstrass M-test, as explained (theorem 1.3 part c) last quarter.
(b) Let \( G = \operatorname{ann}(0; 0, R) \) and let \( f \) be analytic on \( G \) with Laurent series development \( f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \). If \( f_n(z) = \sum_{k=-\infty}^{n} a_k z^k \), show that \( f_n \to f \) in \( C(G; \mathbb{C}) \).

**Solution:** Separating the Laurent series into its positive and negative powers and looking for uniform convergence on smaller annuli, we can reduce to the power series case; see theorem 1.11 for details.