Permutation of Positive Integers Containing no Monotone k-term Arithmetic progression

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Introduction

- ***On Permutation Containing no Long Arithmetic Progression*** by J.A. David, R.C. Entringer, and R. Graham, Acta Arithmetic XXXIV(1977), pages 81-90.

- In this paper they investigate several questions related to existence of monotone *k*-term arithmetic progression, $AP(k)$, in permutation of finite intervals of positive integers and also positive integers.

Example

- Lets consider the set $\{1,2,3,4,5,6\}$. Is there a permutation of this set such that there is no monotone $AP(3)$?

$$\{4,3,5,1,2,6\}$$

$$\{2,4,3,6,5,1\}$$

$$\{3,1,2,5,6,4\}$$

$$\vdots$$

Question

- What about the set $\{1,2,..,n\}$? *Is there a permutation of this set that has no* monotone $AP(3)$?

Yes! There is at least one!

Proof: (by induction)

- Let $A = a_1 a_2 \ldots a_m$ be a permutation of $\{1,2,\ldots,m\}$ with no monotone $AP(3)$.

  Then $2A$ and $2A\text{-}1$ have no monotone $AP(3)$!

  Example

  A: $\{3,1,2,5,6,4\}$

  $2A$: $\{6,2,4,10,12,8\}$

  $2A\text{-}1$: $\{5,1,3,9,11,7\}$

Proof (continued)

- Let

$$(2A)(2A-1)=(2a_1)(2a_2)...(2a_m)(2a_1-1)(2a_2-1)...(2a_m-1)$$

- i.e. $(2A)(2A\text{-}1) = \{6,2,4,10,12,8,5,1,3,9,11,7\}$
- Then $(2A)(2A\text{-}1)$ has no monotone $AP(3)$, and it is a permutation of $\{1,2,..,2m\}$.

$Q.E.D$

Permutation of finite intervals

- We saw that there exist at least one permutation of the set $\{1,2,3\ldots n\}$ such that there is no monotone $AP(3)$.

- Question?

  How many permutations of $\{1,2,3\ldots,n\}$ has no monotone $AP(3)$?

Permutation of finite intervals

- *Thm:* *Let $M(n)$ be the number of permutations* $a_1a_2..a_m$ of $\{1,2,\ldots,n\}$ containing no monotone $AP(3)$ then:

$$M(n) \geq 2^{n-1}, n \geq 1$$

$$M(2n-1) \leq (n!)^2, n \geq 1$$

$$M(2n) \leq (n+1)(n!)^2, n \geq 1.$$

Permutation of finite intervals

n	$M(n)$
1	1
2	2
3	4
4	10
5	20
6	48
7	104
8	282
9	496
10	1066

n	$M(n)$
11	2460
12	6128
13	12840
14	29380
15	74904
16	212728
17	368016
18	659296
19	1371056
20	2937136

Permutation of positive integers

- Let $A = a_1 a_2 a_3 \ldots$ be a permutation of the $\mathbb{Z}^+$, positive integers.

- Let P_k be the set of A's with no monotone $AP(k)$.

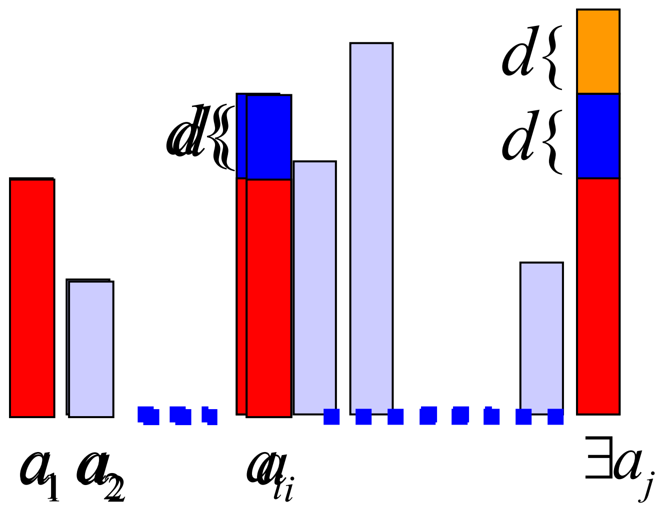
- Then we have the following result:

Permutation of positive integers

$P_3 = \emptyset.$

There is always a monotone *AP(3)* in *A*

Proof: If *i* is the first index such that $a_1 < a_i$ then:



$$a_i = a_1 + d$$

$$a_j = a_1 + 2d$$

Permutation of positive integers

- $P_5 \neq \varnothing$:

- There is at least one permutation of set of positive integers such that it does not contains a monotone $AP(5)$.

Permutation of positive integers

- Set up for the proof: Let

$$A_k = [a_k + 1, a_k + 10^k],$$

$$B_k = [b_k + 1, 2(b_k + 10^k)],$$

$$a_k = 2\sum_{i=0}^{k} (10^i),$$

$$b_k = a_k + 10^k,$$

$$a_0 = 0, b_0 = 0,$$

$$k \geq 0.$$

$$\because \left|A_k\right| = \left|B_k\right| = 10^k, k \geq 0.$$

Permutation of positive integers

- For Example:

$$A_0 = \{1\}, B_0 = \{2\},$$

$$|A_0| = |B_0| = 1$$

$$A_1 = [3, 12], B_1 = [13, 22],$$

$$|A_1| = |B_1| = 10$$

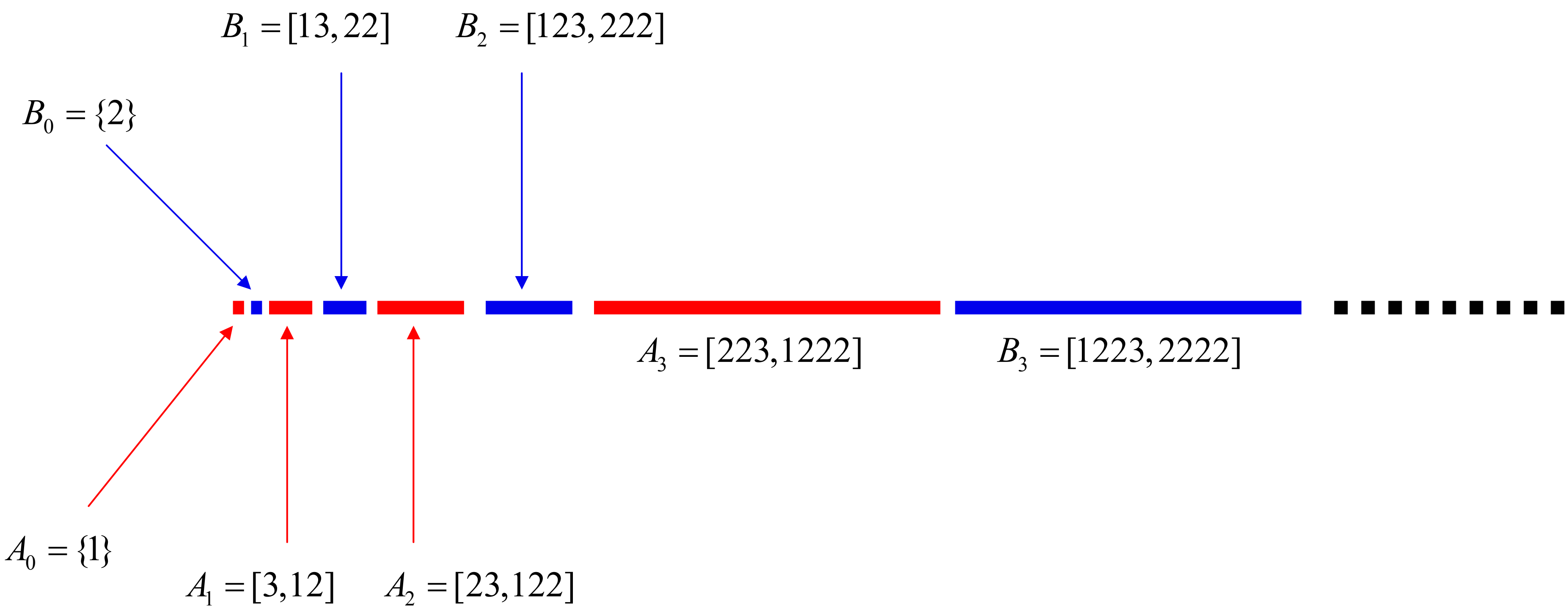
$$A_2 = [23, 122], B_2 = [123, 222],$$

$$|A_2| = |B_2| = 100$$

$$A_3 = [223, 1222], B_3 = [1223, 2222],$$

$$|A_3| = |B_3| = 1000$$

Permutation of positive integers



Permutation of positive integers

- Now let $A_k^* = \pi_k(A)$ and $B_k^* = \sigma_k(B)$ be an arbitrary fixed permutation of A_k and B_k respectively, which contains no monotone $AP(3)$.

  Example:

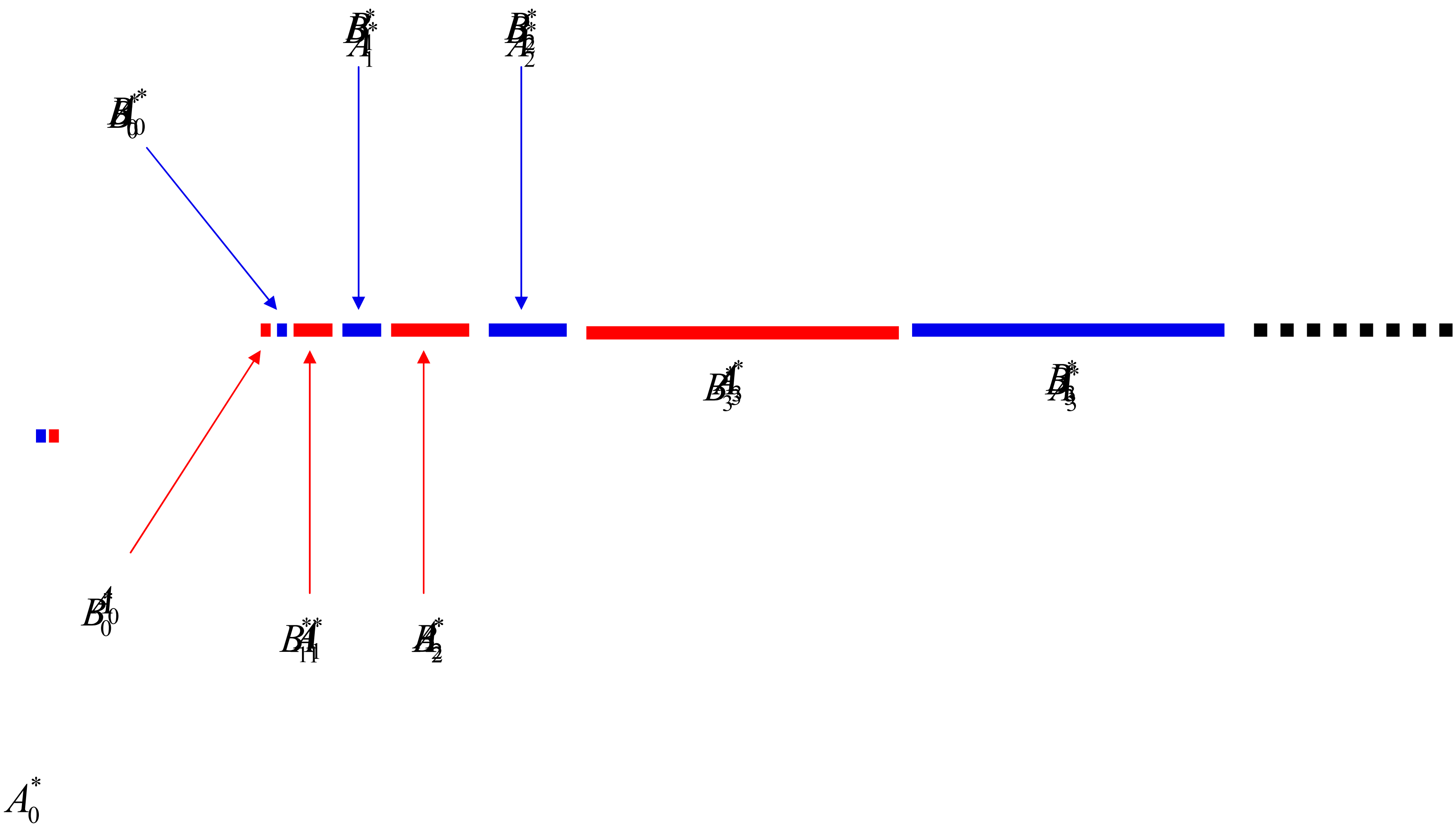
  $$B_1 = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$
  $$B_1^* = 8, 4, 6, 12, 10, 7, 3, 5, 11, 9$$

Permutation of positive integers

- Finally Let I be the permutation of $\mathbb{Z}^+$ such that,

$$I = B_0^* A_0^* B_1^* A_1^* B_2^* A_2^* \ldots B_k^* A_k^* \ldots$$

Permutation of positive integers



Permutation of positive integers

- Claim: I contains no $AP(5)$.
- Proof by contradiction:
  Let

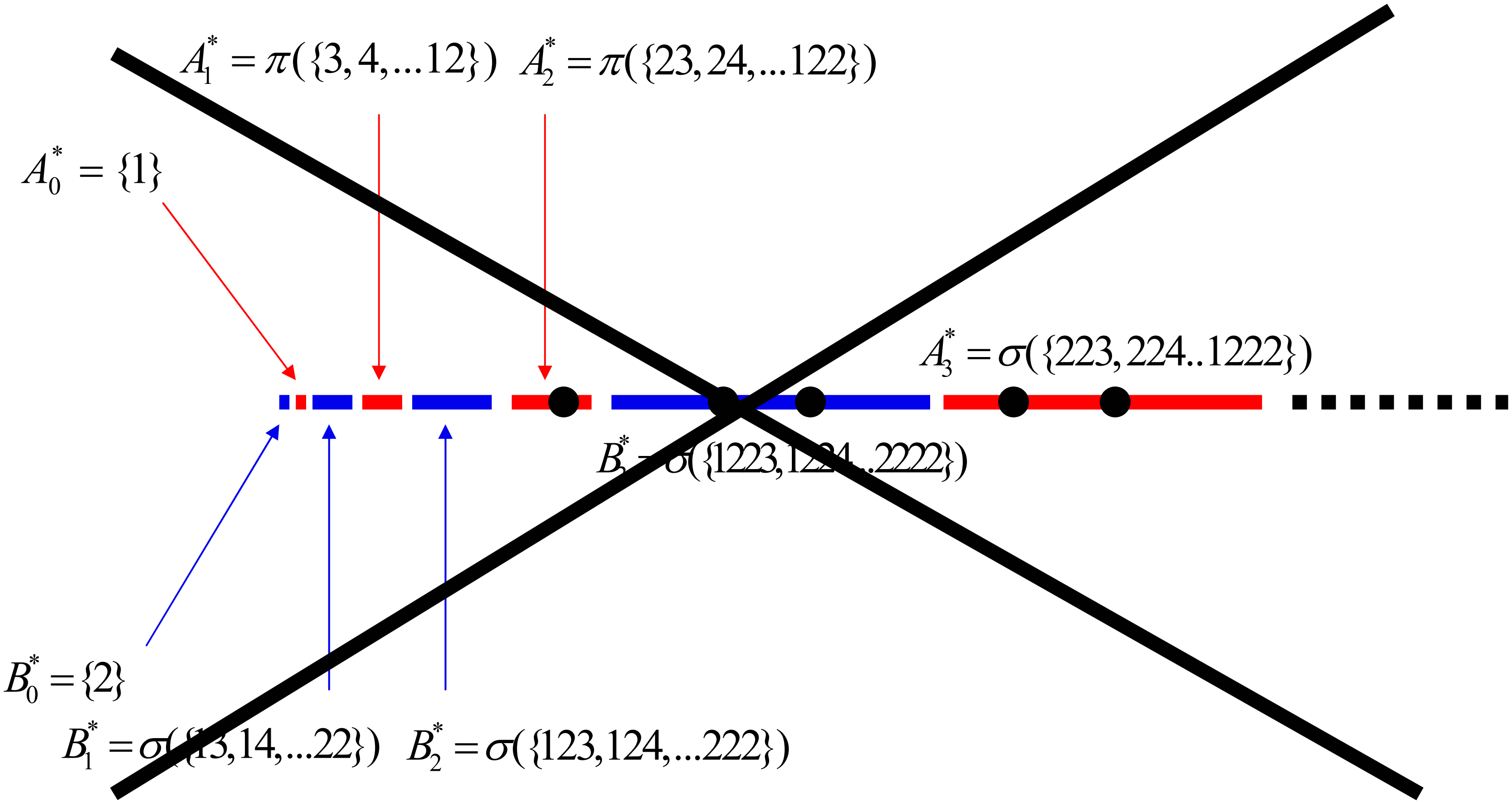
$$X = \{x_1, x_2, x_3, x_4, x_5\},$$

$$\ni x_{k+1} - x_k > d, k \in \{1, 2, 3, 4\},$$

$$X \subset \mathbb{Z}^+$$

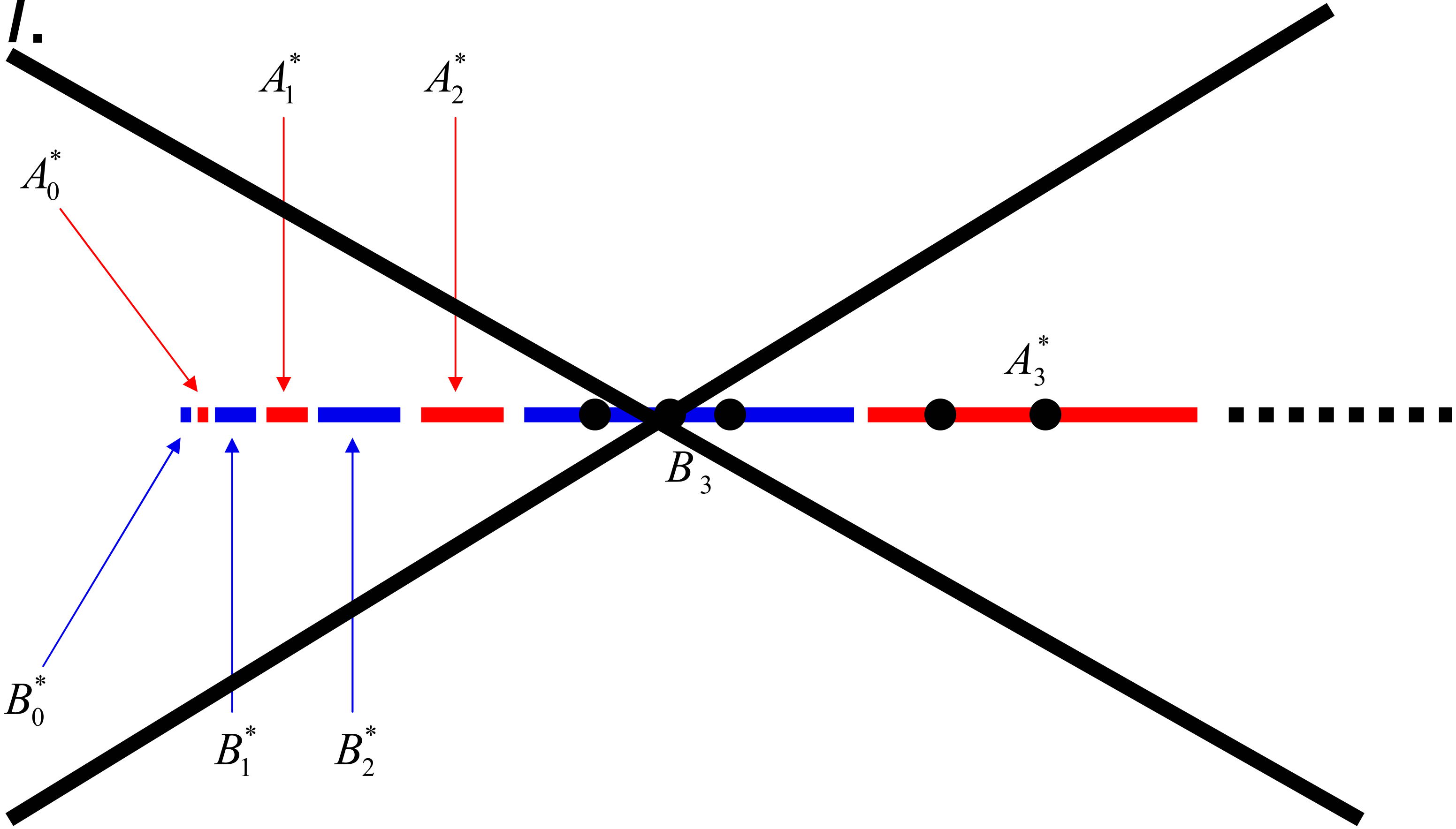
Permutation of positive integers

- $X \subseteq A_k \cup B_k$:



Permutation of positive integers

- Case 1: *X* is a decreasing subsequence of *I*.



Permutation of positive integers

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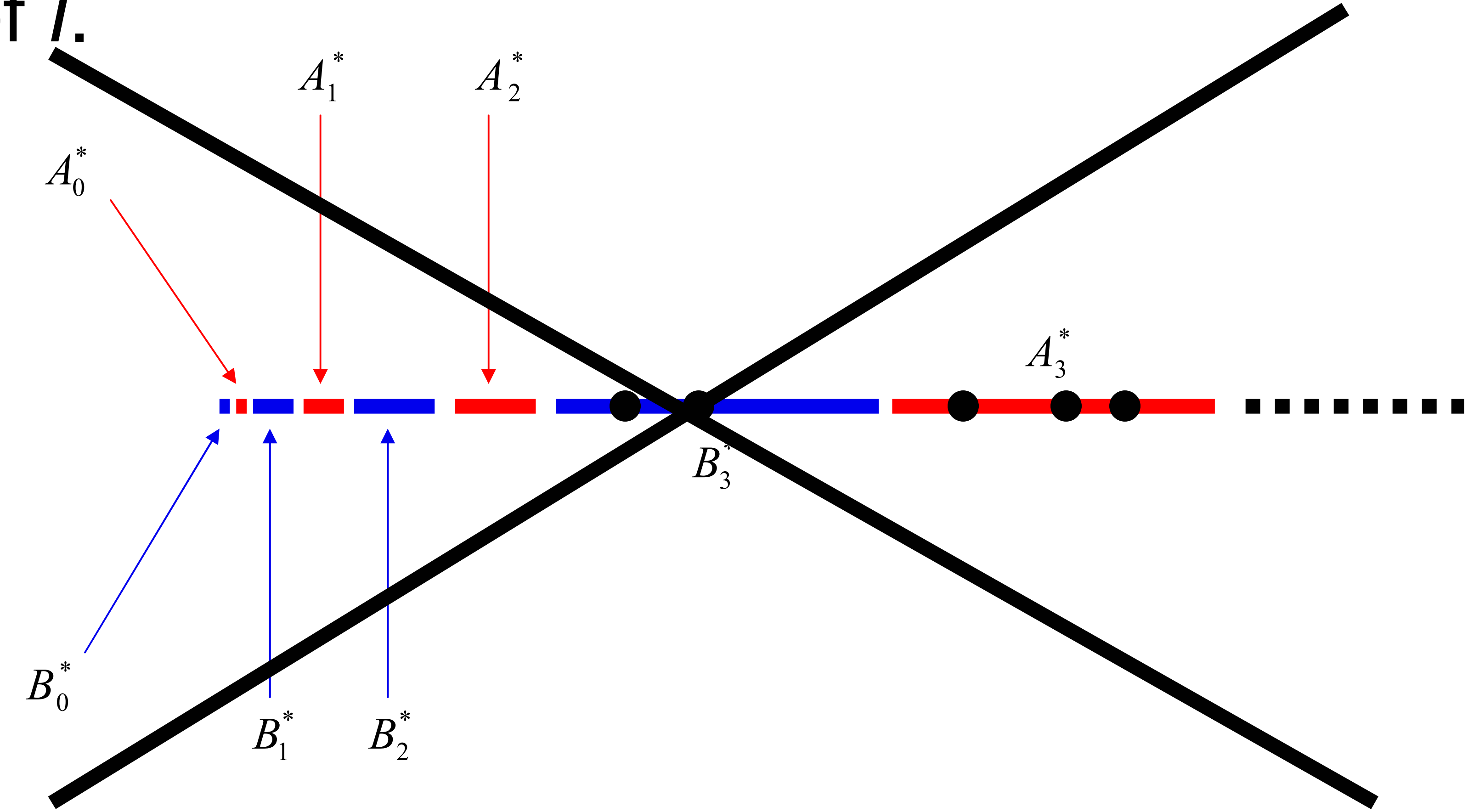

Permutation of positive integers

- Case 2: X is an increasing subsequence of I.
- Proof:

LATER!

Permutation of positive integers

- **Open Problem:**

  What about the P_4?

  Is there (or not) a permutation of $\mathbb{Z}^+$ such that there is (no) monotone $AP(4)$?

Acknowledgments

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Thank you Dr. Brown.

We hope you feel better soon.