Hydrodynamics limits for Randomized Load Balancing

Joint work with Kavita Ramanan

Brown University

November 2014
Model of Interest

Network with
- N servers
- an infinite capacity queue for each server
- a common arrival process
- FCFS service discipline within each queue (no processor sharing)

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Model of Interest

Load Balancing Algorithm:
- How to assign incoming jobs to servers?
- Aim to achieve good performance with low computational cost

Goal: Analysis and comparison of different load balancing algorithms

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Model of Interest

Appears in:
- Supermarkets
- Hash tables
- Distributed memory machines
- Path selection in networks
- Web Servers
- etc.

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Hydrodynamics limits for Randomized Load Balancing
Each arriving job
- chooses $d$ queues out of $N$, uniformly at random,
- joins the shortest queue among the chosen $d$.
- ties broken uniformly at random.
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Supermarket model for exponential service time

Fluid limit and steady state queue length decay rate is obtained

- case $d = 2$, [Vvedenskaya-Dobrushin-Karpelevich ’96]
- case $d \geq 2$, [Mitzenmacher ’01]
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General Approach

Using Markovian state descriptor \{${S}_\ell^N(t); \ell \geq 1, t \geq 0$\}

- \(S_\ell^N(t)\): fraction of stations with at least \(\ell\) jobs
- Convergence as \(N \to \infty\) proved using an extension of Kurtz’s theorem
- The limit process is a solution to a sequence of ODEs
- Steady state queue length distribution is obtained by the fixed point of the ODE sequence

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Hydrodynamics limits for Randomized Load Balancing
Summary of Results

- Joint the Shortest Queue (JSQ)
  - Performance: $P(X_N(\infty) > \ell) \to 0$ for $\ell \geq 1$
  - Computational Cost: $N$ comparison per routing (not feasible)

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Prior Work - Exponential Service Distribution

Summary of Results

- Joint the Shortest Queue (JSQ)
  - Performance: $P(X^N(\infty) > \ell) \rightarrow 0$ for $\ell \geq 1$
  - Computational Cost: $N$ comparison per routing (not feasible)

- $d = 1$ (random routing, decoupled $M/M/1$ queues):
  - Performance: $P(X^N(\infty) > \ell) \rightarrow c\lambda^\ell$
  - Computational cost: one random flip per routing

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Hydrodynamics limits for Randomized Load Balancing
Summary of Results

- **Joint the Shortest Queue (JSQ)**
  - Performance: $P(X^N(\infty) > \ell) \to 0$ for $\ell \geq 1$
  - Computational Cost: $N$ comparison per routing *(not feasible)*

- **$d \geq 2$ (supermarket model):**
  - Performance: $P(X^N(\infty) > \ell) \to \lambda(d^\ell - 1)/(d-1)$
  - Computational Cost: $d$ random flips and $d-1$ comparison per routing

- **$d = 1$ (random routing, decoupled $M/M/1$ queues):**
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  - Computational cost: one random flip per routing

- **Power of two Choices**: double-exponential decay for $d \geq 2$
Our Focus: General service time distribution

- almost nothing was known 5 years ago
- Mathematical Challenge:
  - $\{S^N_\ell\}$ is no longer Markovian
  - need to keep track of more information
  - No finite dimensional common state space for Markovian Representations
Prior Work - General Service Distribution

Recent Progress

- Stability of pre-limit systems [Foss-Chernova’98]
- Tightness of stationary distributions sequence [Bramson’10]
- Stationary queue length decay [Bramson-Lu-Prabhakar’13]

Approach Taken in [Bramson-Lu-Prabhakar’13]:
- Cavity Method
- only proved for service distribution with decreasing hazard rate
- assumes Poisson arrival (uses Poisson splitting)
- only applicable for steady-state distribution
- Pro: can also be easily applied to processor sharing

Joint work with Kavita Ramanan

Hydrodynamics limits for Randomized Load Balancing
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Simulation results for *fraction of busy server*:

- Poisson arrival with $\lambda = 0.5$
- 1000 servers
- Empty initial condition

*Simulation results by Xingjie Li, Brown University
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Hydrodynamics limits for Randomized Load Balancing
Transient Behavior - Simulation

Simulation results for *fraction of busy server*y:

- Poisson arrival with $\lambda = 0.5$
- 1000 servers
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Joint work with Kavita Ramanan
Hydrodynamics limits for Randomized Load Balancing
Simulation results for \textit{fraction of busy server}†

- Poisson arrival with $\lambda = 0.5$
- 1000 servers
- initially one job in each server

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Hydrodynamics limits for Randomized Load Balancing
Simulation results for *fraction of busy server†*

- Poisson arrival with $\lambda = 0.5$
- 1000 servers
- Initially one job in each server

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Hydrodynamics limits for Randomized Load Balancing
Simulation results for *fraction of queues with queue length at least* 2‡

- Poisson arrival with $\lambda = 0.5$
- 1000 servers
- initially one job in each server

‡Simulation results by Xingjie Li, Brown University
Our Goal

Observations:

- No result on the time scale to reach equilibrium
- Transient behavior is also important
- No result on distributions without decreasing hazard rate
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Our Goal:

Introduce a new approach: Interacting Measure-valued Processes
$\nu_{\ell}$: unit mass at the ages of jobs in servers with queues of length at least $\ell$.
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\( \nu_1 \) and at least one jobs.
\[ \nu_\ell: \text{ unit mass at the ages of jobs in servers with queues of length at least } \ell. \]

at least two jobs
$\nu_\ell$: unit mass at the ages of jobs in servers with queues of length at least $\ell$.

at least three jobs
\( \nu_\ell \): unit mass at the ages of jobs in servers with queues of length at least \( \ell \).

at least four jobs
\( \nu_\ell: \) unit mass at the ages of jobs in servers with queues of length at least \( \ell \).

at least five jobs
$\nu_\ell$: unit mass at the ages of jobs in servers with queues of length at least $\ell$.

Analogous to [Kaspi-Ramanan'11]
I. when no arrival/departure is happening, the masses move to the right with unit speed.
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\[ D_\ell: \text{cumulative departure process from servers with at least } \ell \text{ jobs before departure.} \]
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$D_\ell$: cumulative departure process from servers with at least $\ell$ jobs before departure.
III. Upon arrival a queue with $\ell - 1$ jobs right before arrival,

- if $\ell = 1$, a mass at zero joins $\nu_1$
- if $\ell \geq 2$, the mass corresponding to the age of job in that particular server is added to $\nu_\ell$

\[ \nu_{\ell-1} \]

\[ \nu_\ell \]

\[ \nu_{\ell+1} \]

Exactly $\ell-1$ customers
III. Upon arrival a queue with $\ell - 1$ jobs right before arrival,

- if $\ell = 1$, a mass at zero joins $\nu_1$
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\[ \mathcal{R}_\ell : \text{routing measure process} \]
Upon arrival of $j^{th}$ job,

- server $i$ has $\ell$ job: $X^i = \ell$.
- $\zeta_j$ is the index of the server to which job $j$ is routed

what is the probability $\{\zeta_j = i | X^i = \ell\}$?
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$\mathbb{P}\{\text{server } i \text{ has queue length } \geq \ell\} = \mathbb{P}\{\text{all picks have queue length } \geq \ell\} = S^d_{\ell}$.

$$S_{\ell} = \frac{1}{N} \langle 1, \nu_{\ell} \rangle : \text{portion of servers with at least } \leq \ell$$
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2. $\mathbb{P}\{\text{server } \zeta_j \text{ has exactly } \ell \text{ job}\} = S^d_\ell - S^d_{\ell+1}$. 

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3. $\mathbb{P}\{\zeta_j = i|X^i = \ell\} = \frac{1}{N} \frac{S^2_{\ell} - S^2_{\ell+1}}{S_{\ell} - S_{\ell+1}}$
**Definition** A process \( \bar{\nu} = \{ \bar{\nu}_\ell \}_{\ell \geq 0} \) solves the *age equations* if for all \( f \in \mathcal{C}^1_b [0, \infty) \),

\[
\langle f, \nu_\ell (t) \rangle = \langle f, \nu_\ell (0) \rangle
\]
Main Results

**Definition** A process $\bar{\nu} = \{\bar{\nu}_\ell\}_{\ell \geq 0}$ solves the *age equations* if for all $f \in \mathbb{C}^1_b[0, \infty)$,

$$\langle f, \nu_\ell(t) \rangle = \langle f, \nu_\ell(0) \rangle + \int_0^t \langle f', \nu_\ell(s) \rangle ds$$

**Joint work with Kavita Ramanan**

Hydrodynamics limits for Randomized Load Balancing
Definition A process $\tilde{\nu} = \{\tilde{\nu}_\ell\}_{\ell \geq 0}$ solves the age equations if for all $f \in C^1_b[0, \infty)$,

$$\langle f, \nu_\ell(t) \rangle = \langle f, \nu_\ell(0) \rangle + \int_0^t \langle f', \nu_\ell(s) \rangle ds + f(0) D_{\ell+1}(t)$$

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$$

$$
\langle 1, \nu(t) \rangle - \langle 1, \nu(0) \rangle = D_{\ell+1}(t) + \int_0^t \langle 1, \eta_\ell(s) \rangle ds - D_\ell(t),
$$

mass balance
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departure rate
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$$
\eta_\ell(t) = \begin{cases} 
\lambda(1 - \langle 1, \nu_1(t) \rangle^2)\delta_0 & \text{if } \ell = 1, \\
\lambda\langle 1, \nu_{\ell-1}(t) + \nu_\ell(t) \rangle(\nu_{\ell-1}(t) - \nu_\ell(t)) & \text{if } \ell \geq 2.
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$$
Theorem

Let \( \{\nu^{(N)}(t) = (\nu^{(N)}_\ell(t))_\ell; t \geq 0\} \) be the measure-valued representation for the \( N \)-server system with initial condition \( \nu^{(N)}(0) \). If for some \( \nu_\ell(0) \)

1. arrival process \( E^{(N)} \) is a renewal process with rate \( \lambda^N \), and \( \lambda^N / N \to \lambda \),
2. service distribution \( G \) has mean 1 and density \( g \),
3. for every \( \ell \geq 1 \), \( \nu^{(N)}_\ell(0) / N \to \nu_\ell(0) \),

then

\[
\frac{1}{N} \nu^{(N)}_\ell \to \nu_\ell,
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where \( \nu \) is the unique solution to the age equation corresponding to \( \nu(0) \).
**Main Result**

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**Proof sketch.**

- show the tightness of the sequence \( \{\frac{1}{N} \nu^{(N)}\} \).

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Main Result

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- show the tightness of the sequence \( \{ \frac{1}{N} \nu^{(N)} \} \).
- show that every sub-sequential limit solves the age equation.
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\frac{1}{N} \nu^{(N)}_\ell \to \nu_\ell,
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Proof sketch.

- show the tightness of the sequence \( \{1/N \nu^{(N)}\} \).
- show that every sub-sequential limit solves the age equation.
- use the uniqueness theorem for a unique characterization of sub-sequential limits.

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We can partially solve the age equation: for every $f \in C_b[0, \infty)$

$$
\langle f, \nu_\ell(t) \rangle = \langle f(\cdot + t) \frac{\tilde{G}(\cdot + t)}{\bar{G}(\cdot)}, \nu_\ell(0) \rangle + \int_{[0,t]} f(t - s) \tilde{G}(t - s) dD_{\ell+1}(s)
$$

$$
+ \int_0^t \langle f(\cdot + t - s) \frac{\tilde{G}(\cdot + t - s)}{\bar{G}(\cdot)}, \eta_\ell(s) \rangle ds
$$

Equations (1)-(4) are called Hydrodynamics Equations.
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+ \int_0^t \langle f(\cdot + t - s) \frac{\tilde{G}(\cdot + t - s)}{G(\cdot)}, \eta_\ell(s) \rangle ds
\]  

(1)

and

\[
\langle 1, \nu_\ell(t) \rangle - \langle 1, \nu_\ell(0) \rangle = D_\ell(t) + \int_0^t \langle 1, \eta_\ell(s) \rangle ds - D_\ell(t),
\]  

(2)

with

\[
D_\ell(t) = \int_0^t \langle h, \nu_\ell(s) \rangle ds
\]  

(3)

\[
\eta_\ell(t) = \begin{cases} 
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We can partially solve the age equation: for every $f \in C_b[0, \infty)$

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$$+ \int_0^t \langle f(\cdot + t - s) \frac{\bar{G}(\cdot + t - s)}{G(\cdot)}, \eta_\ell(s) \rangle ds$$ (1)

and

$$\langle 1, \nu_\ell(t) \rangle - \langle 1, \nu_\ell(0) \rangle = D_\ell(t) + \int_0^t \langle 1, \eta_\ell(s) \rangle ds - D_\ell(t),$$ (2)

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$$D_\ell(t) = \int_0^t \langle h, \nu_\ell(s) \rangle ds$$ (3)

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Equations (1)-(4) are called Hydrodynamics Equations.
A PDE representation

If one is only interested in $S_\ell(t) = \langle 1, \nu_\ell(t) \rangle$,

$$
\langle 1, \nu_\ell(t) \rangle = \langle \frac{\bar{G}(\cdot + t)}{G(\cdot)}, \nu_\ell(0) \rangle + \int_{[0,t]} \bar{G}(t - s) dD_{\ell+1}(s)
$$

$$
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(5)
A PDE representation

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$$+ \int_0^t \langle \frac{\bar{G}(\cdot + t - s)}{\bar{G}(\cdot)}, \eta_\ell(s) \rangle ds$$

(5)

define

$$f^r(x) = \frac{1 - G(x + r)}{1 - G(x)}, \quad \xi_\ell(t, r) = \langle f^r, \nu_\ell(t) \rangle.$$
A PDE representation

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$$+ \int_0^t \langle \frac{\bar{G}(\cdot + t - s)}{\bar{G}(\cdot)}, \eta_\ell(s) \rangle ds$$

(5)

define

$$f^r(x) = \frac{1 - G(x + r)}{1 - G(x)}$$, \quad \xi_\ell(t, r) = \langle f^r, \nu_\ell(t) \rangle.$$

Then, we have $D_\ell(t) = - \int_0^t \partial_r \xi_\ell(s, 0) ds$, and the PDE
A PDE representation

If one is only interested in $S_\ell(t) = \langle 1, \nu_\ell(t) \rangle$,

$$
\langle 1, \nu_\ell(t) \rangle = \langle \frac{\bar{G}(\cdot + t)}{G(\cdot)}, \nu_\ell(0) \rangle + \int_{[0,t]} \bar{G}(t - s)dD_{\ell+1}(s)
$$

$$
+ \int_0^t \langle \frac{\bar{G}(\cdot + t - s)}{\bar{G}(\cdot)}, \eta_\ell(s) \rangle ds
$$

(5)

define

$$
f^r(x) = \frac{1 - G(x + r)}{1 - G(x)}, \quad \xi_\ell(t, r) = \langle f^r, \nu_\ell(t) \rangle.
$$

Then, we have $D_\ell(t) = -\int_0^t \partial_r \xi_\ell(s, 0)ds$, and the PDE

$$
\xi_\ell(t, r) = \xi_\ell(0, t + r) - \int_0^t \bar{G}(t + r - u)\xi'_{\ell+1}(u, 0)du,
$$

$$
+ \lambda \int_0^t F(\xi_{\ell-1}(u, r), \xi_\ell(u, r))du,
$$
A PDE representation

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\langle 1, \nu_\ell(t) \rangle = \langle \frac{\bar{G}(\cdot + t)}{G(\cdot)}, \nu_\ell(0) \rangle + \int_{[0,t]} \bar{G}(t - s) dD_{\ell+1}(s)
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$$

(5)

define

$$
fr(x) = \frac{1 - G(x + r)}{1 - G(x)}, \quad \xi_\ell(t, r) = \langle fr, \nu_\ell(t) \rangle.
$$

Then, we have $D_\ell(t) = - \int_0^t \partial_r \xi_\ell(s, 0) ds$, and the PDE

$$
\xi_\ell(t, r) = \xi_\ell(0, t + r) - \int_0^t \bar{G}(t + r - u) \xi_\ell'(u, 0) du,
$$

$$
+ \lambda \int_0^t F(\xi_{\ell-1}(u, r), \xi_\ell(u, r)) du,
$$

with boundary condition

$$
\xi_\ell(t, 0) - \xi_\ell(0, 0) = \int_0^t \lambda(u) (\xi_{\ell-1}(u, 0)^2 - \xi_\ell(u, 0)^2) - (\xi_{\ell-1}'(u, 0) - \xi_\ell'(u, 0)^2) du,
$$

Joint work with Kavita Ramanan

Hydrodynamics limits for Randomized Load Balancing
We introduced a framework to analysis the load balancing algorithm, featuring

- Hydrodynamics limit which captures transient behavior
- Applicable for general service distributions
- Applicable for more general time varying arrival processes

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Equilibrium distributions are characterized by the fixed point of the PDEs.
We can numerically solve the PDE and compare to the simulation result we previously saw.
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The PDE provides more efficient alternative to simulations in order to address network optimization and design questions.

- Rate of Convergence Result
- More on Numerical solution for the PDEs
- Gaining insight to specific time-varying scenarios
- Fixed point analysis for the PDE to derive stationary distribution

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Hydrodynamics limits for Randomized Load Balancing
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Our Interacting measure-valued processes framework is general

- Applicable for different load balancing algorithms
- Applied to the analysis of the Serve the Longest Queue (SLQ) service discipline [Ramanan, Ganguly, Robert]

Ongoing Work

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Hydrodynamics limits for Randomized Load Balancing
Discussion and Ongoing Work

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