HOMEWORK 2 SOLUTIONS

Section 2.1

Problem 4

(a) $c = 0$ matches (iii), $c = 1$ matches (ii), and $c = 4$ matches (i)

Problem 5

These drawings can be found in the back of your book.

Problem 15

The level curves are all circles of radius $\sqrt{c}$, centered at the origin.

Problem 18

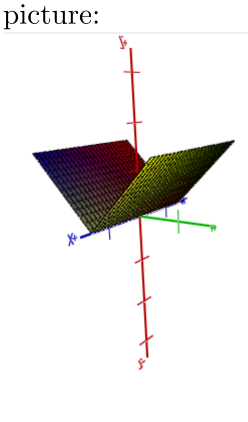
When $c = 0$, the level curve is $x = 0$. For other values of c, graph the line $y = x/c$.

Problem 22

This graph is hard to visualize because it maps $\mathbb{R}^3$ to $\mathbb{R}$, however we can almost think of it as a function from $\mathbb{R}^2$ to $\mathbb{R}$ because it doesn't depend on the z coordinate of input at all. The level set for $c = 0$ consists of points of the form $(0, y, z)$ and $(x, 0, z)$. For other values of c, the level set contains points of the form $(x, c/x, z)$. For the section in the xy plane we simply have $f(x, y, z) = xy$, since setting $z = 0$ doesn't matter. For the section in the xz or yz plane, we have $f(x, y, z) = 0$, since we must set either y or x equal to 0.

Problem 25

The level sets are of the form $|y| = c$, so each will look like a pair of horizontal lines when $c > 0$, will be the x-axis when $c = 0$, and won't exist when $c < 0$. You can visualize this graph by first imagining the graph $z = |y|$ in the yz plane, adding in the x axis, and allowing x to vary. Here is a picture:



Note: I won't ask you to draw 3-dimensional graphs on an exam, but if you're curious how I made this picture and would like to be able to easily visualize functions of two variables, consider downloading GraphCalc. It's a free program that makes 3D graphs and lets you rotate them around to get a good look at the function you're working with.

Problem 26

There are no level curves when $c < 0$. When $c = 0$, we get the x and y axis. When $c > 0$ we get the set of points (x, y) where $y = \pm c$ and $|y| \geq |x|$, in addition to the set of points where $x = \pm c$ and $|x| > |y|$. These look like squares centered at the origin. The section in the xz plane looks like $z = |x|$ and the section in the yz plane looks like $z = |y|$. The graph looks like an upsidedown pyramid centered at the origin.

Section 2.2

I'd like to make a quick comment about computing limits: Some of you have seen a source online which claimes that to compute $\lim_{(x,y)\to(0,0)} f(x,y)$, it's enough to compute the limit along any straight line of approach (ie $y = mx$ or $x = 0$ and verify that all limits exist and agree.) This sounds great at first, but it turns out that even this isn't enough. You'd need to check that the limits agree along any *path* of approach, even nonlinear ones. As a concrete example, consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}.$$

When $x = 0$, the limiting value is 0. When $y = mx$, we have

$$\frac{m^2x^3}{x^2 + m^4x^4}.$$

Two applications of L'Hôpital's rule show that this limit is 0 as well. However, the limit actually doesn't exist. To see this, consider approaching the origin along the line $x = y^2$. Then the function becomes

$$\frac{y^4}{y^{4+4}} = 1/2.$$

The moral of the story: don't believe everything you read on the internet, and be very careful when computing limits.

Problem 3

(a) 0, by plugging in since the function is continuous at $(0, 1)$
(b) -1/2, after two applications of L'Hôpital's rule
(c) 1, by either using L'Hôpital's rule or noticing that this is just the limit definition of the derivative of e^x at $x = 0$.

Problem 4

(a) 1, by plugging in since the function is continuous at $(0, 1)$
(b) 0, by L'Hôpital's rule
(c) 1, by two applications of L'Hôpital's rule.

Problem 6

(a) Along $x = 0$ the limit is 0
(b) Along $x = y^3$ the limit is $1/2$
(c) If f were continuous then its limit along any path of approach to $(0,0)$ would be the same. According to parts (a) and (b), this is not the case.

Problem 8

(a) Simplifying the numerator and canceling gives 4
(b) We use the fact that for any value of x, $|\sin(x)| \leq |x|$. Now we can apply the squeeze theorem:

$$\left|\frac{\sin(xy)}{y}\right| \leq \left|\frac{xy}{y}\right| = |x|.$$

Since $\lim_{x\to 0} |x| = 0$, the original limit must be 0 as well.
(c) First write the numerator as $(x-y)(x^2+xy+y^2)$. Using the fact that $|xy| \leq x^2+y^2$ we have

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| \leq \left|\frac{(x-y)(2x^2+2y^2)}{x+2+y^2}\right| = 2\,|x-y|$$

Since

$$\lim_{(x,y)\to(0,0)} 2|x-y| = 0,$$

the original limit must be 0 as well.

Problem 10

(a) 1, since the function is continuous at $(0,0)$.
(b) The limit does not exist. If we approach the origin along the line $x = 0$, the limit is 0. On the other hand if we approach the origin along the line $y = 0$, the limit is infinite. This can be seen by setting $y = 0$ and then applying L'Hôpital's rule twice.
(c) The limit does not exist. If we approach the origin along the line $y = 0$, the limit is 1. On the other hand, if we set $y = x$, the limit is 0.

Problem 14

The only trouble we run into here is when the denominator is 0. This happens when $x^2+y^2+z^2 = 1$, so the function fails to be continuous on the sphere of radius 1 centered at the origin.

Problem 24

(a) We know that constant functions, x, and $\cos(x)$ are continuous on all of $\mathbb{R}$. The function f is built by taking compositions, products, multiples, and sums of these continuous functions, so the result is continuous.
(b) As in part (a), this function is built from continuous functions by taking sums, products, multiples, and quotients of functions which are continuous everywhere. The only thing to check here is that the denominator is never 0, but that follows because $|\sin(x)| \leq 1$, so we must have $2 - \sin(x) \geq 0$ for all $x \in \mathbb{R}$.

Section 2.3

Problem 3

(a)

$$\frac{\partial w}{\partial x} = 2x^2 e^{x^2+y^2} + e^{x^2+y^2}$$
$$\frac{\partial w}{\partial y} = 2xye^{x^2+y^2}$$

(b)

$$\frac{\partial w}{\partial x} = \frac{-4xy^2}{(x^2-y^2)^2}$$
$$\frac{\partial w}{\partial y} = \frac{4yx^2}{(x^2-y^2)^2}$$

(c)

$$\frac{\partial w}{\partial x} = \frac{2xe^{xy}}{x^2+y^2} + ye^{xy}\log(x^2+y^2)$$
$$\frac{\partial w}{\partial y} = \frac{2ye^{xy}}{x^2+y^2} + xe^{xy}\log(x^2+y^2)$$

(d)

$$\frac{\partial w}{\partial x} = 1/y$$
$$\frac{\partial w}{\partial y} = -x/y^2$$

(e)

$$\frac{\partial w}{\partial x} = \cos(ye^{xy})\cos(x) - y^2 e^{xy}\sin(ye^{xy})\sin(x)$$
$$\frac{\partial w}{\partial y} = -\sin(ye^{xy})(xye^{xy} + e^{xy})\sin(x)$$

Problem 5

First we compute the partial derivatives at the specified point:

$$\frac{\partial z}{\partial x}(3,1) = 6$$

and

$$\frac{\partial z}{\partial y}(3,1) = 3.$$

Putting these together, the equation of the tangent plane is

$$z = 10 + 6(x-3) + 3(y-1).$$

Problem 6

First we compute the partial derivatives at the specified point:

$$\frac{\partial f}{\partial x}(0,0) = 1$$

and

$$\frac{\partial f}{\partial y}(0,0) = 1.$$

Putting these together, the equation of the tangent plane is

$$z = 1 + x + y.$$

Problem 12

(a) First we compute the partial derivatives at the specified point:

$$\frac{\partial f}{\partial x}(0,0) = 2$$

and

$$\frac{\partial f}{\partial y}(0,0) = 3.$$

Putting these together, the equation of the tangent plane is

$$z = 1 + 2x + 3y.$$

(b) $f(.1, 0) \approx 1.2$ and $f(0, .1) \approx 1.3$

(c) It's a bit silly to ask for exact values because calculators approximate as well, but we can give better approximations with the calculator which show that ours were pretty good to begin with:

$$f(.1, 0) \approx 1.2214 \text{ and } f(0, .1) \approx 1.3499.$$

Problem 14

Their tangent planes both have the equation $z = 0$ at that point.

Problem 16

(a) Let $f(x, y) = (xe^y)^8$. We can use a linear approximation at $(x, y) = (1, 0)$. The tangent plane at $(1, 0)$ has equation

$$z = 1 + 8(x - 1) + 8y.$$

Now we plug in $(x, y) = (.99, .02)$ to obtain the approximation

$$f(.99, .02) \approx 1.08.$$

(b) Let $f(x, y) = x^3 + y^3 - 6xy$. The tangent plane at $(1, 2)$ has equation

$$z = -3 - 9(x - 1) + 6(y - 2).$$

Now we plug in $(x, y) = (.99, 2.01)$ to obtain the approximation

$$f(.99, 2.01) \approx -2.85$$

(c) Let $f(x, y) = \sqrt{2x^2 + y^2}$. The tangent plane at $(4, 2)$ has equation

$$z = 6 + (4/3)(x - 4) + (1/3)(y - 2).$$

You might be wondering which value of x makes sense to plug in here, since we see 4.01 and 3.98 each appearing in the problem. One way to handle this is to split the difference and plug in 3.95. Note: this isn't the only way to solve this problem! Approximations, by definition, leave you with a bit of wiggle room. You could also let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and find a linear approximation at $(4, 4, 2)$, then plug in values to that.

$$f(3.95, 2.02) \approx 5.94.$$

Problem 24

$$\nabla h = ((x+z)e^{x-y} + e^{x-y}, -(x+z)e^{x-y}, e^{x-y})$$

so we have

$$\nabla h(1,1,1) = (3,-2,1).$$

Problem 26

$$\nabla f = \left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2}\right)$$

so we have

$$\nabla f(1,0,1) = (1,0,1).$$

Section 2.4

Problem 1

The curve is an ellipse centered at the origin with width 1 and height 4.

Problem 2

The curve is an ellipse centered at the origin with width 2 and height 4.

Problem 5

(a) $c(t) = (2\cos(t), 2\sin(t))$, $t \in [0, 2\pi]$.
(b) $c(t) = (2\sin(t), 2\cos(t))$, $t \in [0, 2\pi]$.
(c) $c(t) = (4 + 2\cos(t), 7 + 2\sin(t))$, $t \in [0, 2\pi]$.

Problem 8

The velocity vector is $c'(t) = (3\cos(3t), -3\sin(3t), 3\sqrt{t})$.

Problem 12

The tangent vector is $c'(t) = (6t, 3t^2)$.

Problem 16

The general velocity vector is $c'(t) = (6, 6t, 3t^2)$. At $t = 0$, the velocity vector is $(6,0,0)$.

Problem 18

First we compute $c'(t) = (-2\cos(t)\sin(t), 3 - 3t^2, 1)$. At $t = 0$ this is $(0,3,1)$, which gives us our direction vector. We also compute $c(0) = (1,0,0)$. An equation of the tangent line to the curve at $t = 0$ is

$$\ell(t) = (1,0,0) + t(0,3,1).$$

Problem 20

First we compute the velocity vector: $c'(t) = (e^t, -e^{-t}, -\sin(t))$. When $t = 1$ this is $(e, -e^{-1}, -\sin(1))$. After 1 unit of time, the particle travels along the vector $(e, -e^{-1}, -\sin(1))$. Finally, we need to add that to the initial position of the particle. This is found by computing $c(1) = (e, e^{-1}, \cos(1))$. Thus, the position after 2 seconds is:

$$(2e, 0, -\sin(1) + \cos(1)).$$