

HOMEWORK 3 SOLUTIONS

Section 2.5

Problem 3

- (a) I'll give a detailed solution for part (a), and just give the derivative for the remaining parts. This problem is asking you to compute the same thing in two different ways. First we'll use the chain rule for paths:

$$\nabla f(x, y) = (y, x) \text{ so } \nabla f(c(t)) = (\cos(t), e^t).$$

Then we compute $c'(t) = (e^t, -\sin(t))$. Taking their dot product, we have:

$$\frac{d}{dt}f(c(t)) = e^t \cos(t) - e^t \sin(t).$$

On the other hand, we could have first computed their composition:

$$f(c(t)) = e^t \cos(t).$$

Then we have

$$\frac{d}{dt}f(c(t)) = e^t \cos(t) - e^t \sin(t)$$

by using the normal product rule for functions of a single variable. Either way we compute, we get the same result.

- (b)

$$\frac{d}{dt}f(c(t)) = 15t^4 e^{3t^5}.$$

- (c)

$$\frac{d}{dt}f(c(t)) = (e^{2t} - e^{-2t})(1 + \log(e^{2t} + e^{-2t})).$$

- (d)

$$\frac{d}{dt}f(c(t)) = (1 + 4t^2)e^{2t^2}.$$

Problem 7

First we compute the composition:

$$f(g(x, y)) = (\tan(e^{x-y} - 1) - e^{x-y}, e^{2x-2y} - (x - y)^2).$$

Now we compute the matrix of partial derivatives:

$$\begin{bmatrix} \sec^2(e^{x-y} - 1) - e^{x-y} & -\sec^2(e^{x-y} - 1) + e^{x-y} \\ 2e^{2x-2y} - 2(x - y) & -2e^{2x-2y} + 2(x - y) \end{bmatrix}.$$

Finally, we evaluate this at $(1, 1)$:

$$\begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}.$$

Problem 10

- (a) $T(\sigma(t)) = t^2$ so $T'(t) = 2t$.

- (b) We'll use a linear approximation for $T(t)$ when $t = \pi/2$. We know $T(\pi/2) = (\pi/2)^2$ and $T'(\pi/2) = \pi$, so $L(t) = (\pi/2)^2 + (t - \pi/2)(\pi/2)$ is the equation of the tangent line to T at $t = \pi/2$. Thus,

$$T(\pi/2 + .01) \approx (\pi/2)^2 + .01(\pi/2).$$

Problem 13

- (a) First we compute $\nabla f = (2xe^y - y^3, x^2e^y - 3xy^2)$, so that

$$\nabla f(c(t)) = (2\cos(t)e^{\sin(t)} - \sin^3(t), \cos^2(t)e^{\sin(t)} - 3\cos(t)\sin^2(t)).$$

Next we compute $c'(t) = (-\sin(t), \cos(t))$. By the chain rule for paths,

$$\frac{dT}{dt} = -2\cos(t)\sin(t)e^{\sin(t)} + \sin^4(t) + \cos^3(t)e^{\sin(t)} - 3\cos^2(t)\sin^2(t).$$

- (b) First we compose:

$$f(c(t)) = \cos^2(t)e^{\sin(t)} - \cos(t)\sin^3(t).$$

When we differentiate this function with respect to t , we obtain the same result as in part (a).

Problem 15

This problem asks you to compute $\frac{d}{dt}f(c(t))$ when $t = 0$. By the chain rule,

$$\left. \frac{d}{dt}f(c(t)) \right|_{t=0} = Df(c(0)) \cdot c'(0).$$

We have

$$Df = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x-y} & -e^{x-y} \end{bmatrix}$$

so that

$$Df(c(0)) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Since $c'(0)$ is the column vector $(1,1)$, we multiply the matrices together to obtain the column vector $(2,0)$.

Problem 33

This is chain rule for paths at its finest! We don't know the definitions of the underlying function f or the path c , but with the limited information we're provided, we can still compute the derivative as follows:

$$\left. \frac{d(f \circ c)}{dt} \right|_{t=\pi} = \nabla f(c(\pi)) \cdot c'(\pi) = (0, 1, 3, -7) \cdot (19, 11, 0, 1) = 4.$$

Problem 34

- (a) We must have $q = n$ so that the outputs of g are the same dimension as the inputs for f .
 (b) We must have $m = p$ so that the outputs of f are the same dimension as the inputs for g .
 (c) This only makes sense when $n = m$, because the outputs of f need to work for inputs to f .

Problem 35

Let $u(x, y) = x - y$. By a nicer, simpler version of the second special case of the chain rule, we have

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} = -\frac{\partial f}{\partial u}.$$

Adding these together, we have

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

Section 2.6**Problem 3**

(a) $\nabla f = (yx^{y-1}, x^y \ln(x))$ so we have $\nabla f(e, e) = (e^e, e^e)$. Since $\|d\| = 13$, the directional derivative is

$$(e^e, e^e) \cdot (5/13, 12/13) = \frac{17}{13}e^e.$$

(b) $\nabla f = (e^x, z, y)$ so we have $\nabla f(1, 1, 1) = (e, 1, 1)$. Since $\|d\| = \sqrt{3}$, the directional derivative is

$$(e, 1, 1) \cdot (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}) = e/\sqrt{3}.$$

(c) $\nabla f = (yz, xz, xy)$ so we have $\nabla f(1, 0, 1) = (0, 1, 0)$. Since $\|d\| = \sqrt{2}$, the directional derivative is

$$(0, 1, 0) \cdot (1/\sqrt{2}, 0, -1/\sqrt{2}) = 0.$$

Problem 4

To stay at the same level, we need to move orthogonally to the gradient. Since

$$\nabla f = (-\pi y \sin(\pi x) - \cos(\pi y), \cos(\pi x) + \pi x \sin(\pi y))$$

we have $\nabla f(2, 1) = (1, 1)$, so we need to move in a direction which is orthogonal to $(1, 1)$. One possibility is to move in the $(1, -1)$ direction. The other option would be the $(-1, 1)$ direction.

Problem 5

(a) We know that the directional derivative is maximized in the direction of the gradient, so we'll compute it where our direction vector is a unit vector in the direction of the gradient:

$$\nabla f(x_0) \cdot \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|} = \frac{\|\nabla f(x_0)\|^2}{\|\nabla f(x_0)\|} = \|\nabla f(x_0)\|.$$

(b) $\nabla f = (3x^2, -3y^2, 3z^2)$ so $\nabla f(1, 2, 3) = (3, -12, 27)$. By part (a), the maximum value for the directional derivative is $\|(3, -12, 27)\| = \sqrt{882}$.

Problem 6

We need a vector normal to the level curve of the function $f(x, y) = x^3 + xy + y^3$ at the point $(x, y) = (1, 2)$. This is just the gradient at that point: $\nabla f = (3x^2 + y, x + 3y^2)$ so $\nabla f(1, 2) = (5, 13)$.

Problem 8

- (a) To write down the equation of a tangent plane, we simply need a point and a direction vector. The points are all supplied for us, and the direction vector will be the gradient of the function evaluated at the point in question. Let $f(x, y, z) = x^2 + 2y^2 + 3xz$. Then $\nabla f = (2x + 3z, 4y, 3x)$ so $\nabla f(1, 2, 1/3) = (3, 8, 3)$. Thus, an equation for the tangent plane is $(3, 8, 3) \cdot (x - 1, y - 2, z - 1/3) = 0$.
- (b) Let $f(x, y, z) = y^2 - x^2$. Then $\nabla f = (-2x, 2y, 0)$ so $\nabla f(1, 2, 8) = (-2, 4, 0)$. Thus, an equation for the tangent plane is $(-2, 4, 0) \cdot (x - 1, y - 2, z - 8) = 0$.
- (c) Let $f(x, y, z) = xyz$. Then $\nabla f = (yz, xz, xy)$ so $\nabla f(1, 1, 1) = (1, 1, 1)$. Thus, an equation for the tangent plane is $(1, 1, 1) \cdot (x - 1, y - 1, z - 1) = 0$.

Problem 24

Consider a sphere centered at the origin and let \mathbf{x} be a point on that sphere. The nice thing about spheres is that when viewed as a vector, \mathbf{x} is normal to the tangent plane to the sphere at \mathbf{x} . Similarly, we know that $\nabla f(\mathbf{x}) = g(\mathbf{x})\mathbf{x}$, so the gradient is parallel to \mathbf{x} . Thus, the gradient is also orthogonal to the tangent plane to the sphere at \mathbf{x} . The gradient points orthogonally to level surfaces, so we know that the tangent plane to the sphere and the tangent plane to the level surface coincide. Since this holds true for every point on the sphere, the sphere must itself be the level surface, so f is constant on spheres centered at the origin.

If this answer seems confusing or hard to visualize, consider the following more concrete approach: We know that $x^2 + y^2 + z^2 = k$ defines a sphere centered at the origin of radius \sqrt{k} . Let $c(t)$ be any path which lies in that sphere. In other words, if we write $c(t) = (x(t), y(t), z(t))$ then this must satisfy $x(t)^2 + y(t)^2 + z(t)^2 = k$. Using the chain rule for paths, we'll compute $\frac{d}{dt}f(c(t))$. If the result is 0, then we know that f is constant inside the sphere.

$$\begin{aligned} \frac{d}{dt}f(c(t)) &= \nabla f(c(t)) \cdot c'(t) = g(c(t))(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t)) \\ &= g(c(t))(x(t)x'(t) + y(t)y'(t) + z(t)z'(t)). \end{aligned}$$

Remember that $x(t)^2 + y(t)^2 + z(t)^2 = k$. Differentiating both sides of the equation with respect to t , and using the regular chain rule for functions of one variable, we have $2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$. Diving both sides by 2 and substituting into our expression for $\frac{d}{dt}f(c(t))$, we see that the derivative is 0.

Problem 26

The altitude increases most rapidly in the direction of the gradient. $\nabla f = (-2ax, -2ay)$ so $\nabla f(1, 1) = (-2a, -2b)$. If a marble were released at $(1, 1)$, it would roll in the direction of steepest descent, which is opposite the gradient. Thus, the marble would roll in the direction $(2a, 2b)$.

Problem 30

- (a) First write $h(t) = f(c(t)) = \cos(t)\sin(t)$. Now we're simply finding extrema of a function of one variable on the interval $[0, 2\pi]$. Computing the derivative, we have $\frac{d}{dt}h(t) = \cos^2(t) - \sin^2(t)$. This is equal to 0 when $\cos(t) = \pm\sin(t)$, which happens at $t = \pi/4, 3\pi/4, 5\pi/4, \text{ and } 7\pi/4$. We need to check the value of $h(t)$ at each of these critical points, along with the endpoints:

$$\begin{aligned}
 h(0) &= 0 \\
 h(\pi/4) &= 1/2 \\
 h(3\pi/4) &= -1/2 \\
 h(5\pi/4) &= 1/2 \\
 h(7\pi/4) &= -1/2 \\
 h(2\pi) &= 0.
 \end{aligned}$$

Thus, the maximum value attained along the path is $1/2$ and the minimum value is $-1/2$.

- (b) As in part (a), let $h(t) = f(c(t)) = \cos^2(t) + 4\sin^2(t)$. Then $\frac{dh}{dt} = -2\cos(t)\sin(t) + 8\sin(t)\cos(t) = 6\cos(t)\sin(t)$. This is equal to 0 when either $\sin(t)$ or $\cos(t)$ is 0, so when $t = 0, \pi/2, \pi, 3\pi/2,$ and 2π . Since the endpoints are included in this list we don't need to worry about adding them separately. Now we check the value of $h(t)$ at each of these points;

$$\begin{aligned}
 h(0) &= 1 \\
 h(\pi/2) &= 4 \\
 h(\pi) &= 1 \\
 h(3\pi/2) &= 4 \\
 h(2\pi) &= 1.
 \end{aligned}$$

Thus, the maximum value attained along the path is 4 and the minimum value attained is 1.

Section 3.1

Problem 2

$$\begin{aligned}
 f_x &= -1/x^2 + e^{-y} \\
 f_y &= -xe^{-y} \\
 f_{xx} &= 2/x^3 \\
 f_{xy} &= -e^{-y} \\
 f_{yx} &= -e^{-y} \\
 f_{yy} &= xe^{-y}
 \end{aligned}$$

so we have equality of mixed partials.

Problem 6

$$\begin{aligned}
 f_x &= \frac{1}{x-y} \\
 f_y &= \frac{-1}{x-y} \\
 f_{xx} &= \frac{-1}{(x-y)^2} \\
 f_{xy} &= \frac{1}{(x-y)^2} \\
 f_{yx} &= \frac{1}{(x-y)^2} \\
 f_{yy} &= \frac{-1}{(x-y)^2}
 \end{aligned}$$

so we have equality of mixed partials.

Problem 9

If such a C^2 function existed then by Theorem 1 the mixed partial derivatives would be equal. However $f_{xy} = -5$ and $f_{yx} = 4$, so no such function can exist.

Problem 10

$u_t = -ke^{-kt} \sin(x)$, $u_x = e^{-kt} \cos(x)$, and $ku_{xx} = -ke^{-kt} \sin(x)$ so $u_t = ku_{xx}$ and we conclude that $u(x, t)$ satisfies the heat conduction equation.

Problem 20

$$\begin{aligned}
 f_{xzw} &= \frac{\partial}{\partial w} \frac{\partial}{\partial z} \frac{\partial}{\partial x} f \\
 &= \frac{\partial}{\partial w} \frac{\partial}{\partial x} \frac{\partial}{\partial z} f \text{ by Theorem 1} \\
 &= \frac{\partial}{\partial x} \frac{\partial}{\partial w} \frac{\partial}{\partial z} f \text{ since } \frac{\partial f}{\partial z} \text{ is of class } C^2 \\
 &= f_{zwx}.
 \end{aligned}$$

Note that since f is of class C^3 , each of its partial derivatives is of class C^2 . That's how we justified the second to last inequality.

Problem 25

$$\begin{aligned}
 u_x &= 3x^2 - 3y^2 \\
 u_{xx} &= 6x \\
 u_y &= -6xy \\
 u_{yy} &= -6x
 \end{aligned}$$

so we conclude that $u(x, y)$ is harmonic, since $u_{xx} + u_{yy} = 6x - 6x = 0$.

Section 3.3**Problem 4**

To find critical points, we compute both partial derivatives and set them equal to zero: $f_x = 2x + 3y$ and $f_y = 2y + 3x$. These both equal zero only at $(0, 0)$, so that is our only critical point. Now we compute the discriminant to determine what type of point it is:

$$\begin{aligned}f_{xx} &= 2 \\f_{xy} &= 3 \\f_{yy} &= 2\end{aligned}$$

so we have $D(0, 0) = 4 - 3^2 = -5$. Thus $(0, 0)$ is a saddle point.

Problem 10

We set $f_x = \sin(y) = 0$ and $f_y = 1 + x \cos(y) = 0$. By the first equation, $y = k\pi$ where k is an integer. By the second equation, $x = 1$ if k is odd and $x = -1$ if k is even. Now we compute the discriminant:

$$\begin{aligned}f_{xx} &= 0 \\f_{xy} &= \cos(y) \\f_{yy} &= -x \sin(y)\end{aligned}$$

so we have $D(x, y) = -\cos(y)^2$. This is always negative, so every critical point is a saddle point.

Problem 22

First we'll compute the critical points of f : $f_x = 2x + ky = 0$ and $f_y = 2y + kx = 0$. By the first equation, $x = -ky/2$, so by the second equation $4y = k^2y$. First suppose $k \neq \pm 2$. Then we must have $y = 0$, so $(0, 0)$ is a critical point. We have $D = f_{xx}f_{yy} = f_{xy}^2 = 4 - k^2$, so if $|k| > 2$, the origin is a saddle, and if $|k| < 2$ then the origin is a local min, since $f_{xx} = 2 > 0$. When $k = \pm 2$, the second derivative test is inconclusive, and the graph has critical points wherever $y = \mp x$. One can check by hand that these are all local minimums. In summary, the shape of the graph changes qualitatively at $k = -2$ and $k = 2$.

Problem 26

This problem doesn't make sense, so we'll use the corrected version where $f(x, y) = ax^2 + by^2$.

- (a) We compute $f_x = 2ax = 0$ and $f_y = 2ay = 0$, so we must have $x = 0$ and $y = 0$. Thus $(0, 0)$ is the only critical point.
- (b) The discriminant is $4ab$. If a and b are both positive then the origin is a local minimum. If a and b are both negative then the origin is a local maximum. If a and b have different signs then the origin is a saddle point.

Problem 30

Let x , y , and z denote the side lengths of the parallelepiped. Then the volume is given by $V(x, y, z) = xyz$. The surface area is given by $S(x, y, z) = 2xy + 2xz + 2yz$. Since this is constant, we have $S(x, y, z) = c$ for some constant c . Then we can solve $z = (c - 2xy)/(2x + 2y)$ so we can rewrite volume as a function of two variables:

$$V(x, y) = xy(c - 2xy)/(2x + 2y).$$

To find the maximum, we compute the partial derivatives:

$$V_x = \frac{2cy^2 - 8xy^3 - 4x^2y^2}{(2x + 2y)^2} = 0$$

$$V_y = \frac{2cx^2 - 8yx^3 - 4x^2y^2}{(2x + 2y)^2} = 0$$

Clearing denominators and factoring out y^2 and x^2 respectively from the two equations, we have $2c - 8xy - 4x^2 = 0 = 2c - 8xy - 4y^2$. Thus, $x^2 = y^2$. Since side lengths are always nonnegative, this implies $x = y$. Since all variables in the problem are treated equally, this must imply that $x = z$ and $y = z$. If you don't like that explanation, we can simply continue the computation to verify that $x = z$ as follows: By the quadratic formula we have

$$x = \frac{8y \pm \sqrt{64y^2 + 32c}}{-8} = -x \pm \frac{\sqrt{64x^2 + 32c}}{-8}$$

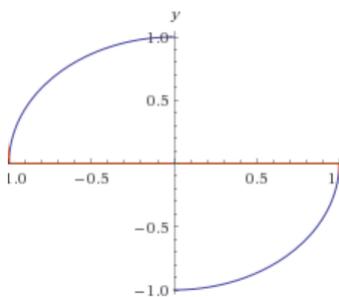
Solving this equation for c gives $c = 6x^2$. Now we have $z = (c - 2xy)/(2x + 2y) = (6x^2 - 2x^2)/(4x) = x$ as desired. Thus, $x = y = z$ so the parallelepiped which maximizes volume for fixed surface area must be a cube.

Problem 40

Rather than minimizing distance, we'll minimize distance squared, because it will give the same solution and make the computations nicer. (This is similar to example 8 in the book). Begin by writing $z = \pm\sqrt{x^2 + y^2 - xy - 1}$. The square of the distance from the origin to a point on this graph is given by

$$x^2 + y^2 + z^2 = x^2 + y^2 + x^2 + y^2 - xy - 1 = 1 - xy.$$

At this point you'd compute partials and determine that $1 - xy$ has a critical point at $(0, 0)$. That's true, but it doesn't lie on the curve. The important part is to realize that $x^2 + y^2 = 1$. ie, we're looking for a minimum on a bounded domain, so we need to worry about what happens on the curve. Furthermore, we have $z^2 = x^2 + y^2 - xy - 1 = -xy$, which implies that x and y have opposite signs, so we only care about (x, y) in the second and fourth quadrant. It's tempting to set $y = \pm\sqrt{1 - x^2}$, but remember which quadrants we're in! The points we care about actually lie on a curve which looks like this:



We can transform the problem into minimizing $1 - xy = 1 - \pm x\sqrt{1 - x^2}$, keeping our restricted domain in mind. This is now a problem from single variable calculus! The function has a critical point at $x = \pm 1/\sqrt{2}$, so we have $y = \pm 1/\sqrt{2}$ as well. However, we have to be careful here. We need to worry about the behavior at the endpoints so we must consider what happens when $x = -1, 0$, or 1 . In these cases, $y = 0, \pm 1$, and 0 respectively. Thus, the critical points are:

$$(1/\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2})$$

and

$$(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0).$$

When we plug these back into the distance formula, we see that distance from each of the first four points to the origin is $\sqrt{3/2}$ and the distance from the second four points to the origin is 1, so ultimately the distance is minimized at the second set of four points, and is equal to 1.

Problem 42

This problem has two parts. We need to search for critical points in the plane which lie inside the given rectangle, and we need to check points on the boundary of the rectangle. First we compute the partial derivatives and set them equal to 0. $f_x = y = 0$ and $f_y = x = 0$ so $(0, 0)$ is a critical point. Now we consider the boundary. The upper part of the box has equation $y = 1$ and $-1 \leq x \leq 1$. Here we don't even need to use calculus: the function is so nice that it's easy to tell it's maximized at $(1, 1)$ and minimized at $(-1, 1)$. By the same reasoning, we see that all maximums and minimums on the boundary occur at the endpoints, so we just need to check the function values at the vertices of the square and the critical point we found at the origin:

$$\begin{aligned} f(0, 0) &= 0 \\ f(-1, -1) &= 1 \\ f(-1, 1) &= -1 \\ f(1, 1) &= 1 \\ f(1, -1) &= -1 \end{aligned}$$

so we conclude that the global maximum and minimum values for f are 1 and -1.

Problem 44

We begin with a search for critical points in the plane. $f_x = y + 1$ and $f_y = x - 2$ so $(2, -1)$ is a critical point, and it lies inside T . Next we check for extrema on the boundary. The bottom boundary of the triangle is given by $y = -2$ where $1 \leq x \leq 5$. We have $f(x, -2) = 5 - x$ which is minimized at $x = 1$ and maximized at $x = 5$. The left vertical boundary of the triangle is given by $x = 1$ where $-2 \leq y \leq 2$. We have $f(1, y) = 2 - y$ which is minimized at $y = 2$ and maximized at $y = -2$. The last side of the triangle is given by $y = -x + 3$. We have $f(x, -x + 3) = -x^2 + 6x - 5$. To find critical points along this boundary, we compute the derivative of this function of one variable, which is $-2x + 6$, so $x = 3$ is a critical point. Thus, $(3, 0)$ is a critical point of the original function. Lastly, we compute the function values at all of these point where absolute extrema may occur:

$$\begin{aligned} f(2, -1) &= 3 \\ f(1, 2) &= 0 \\ f(1, -2) &= 4 \\ f(5, -2) &= 0 \\ f(3, 0) &= 4 \end{aligned}$$

The global (absolute) maximum of f is 4, which occurs at $(1, -2)$ and $(3, 0)$, and the global (absolute) minimum of f is 0, which occurs at $(1, 2)$ and $(5, -2)$.