

Math 20C
Summer II 2016
Exam 2
08/23/2016
Time Limit: 60 Minutes

Name: _____

This exam contains 5 questions. Be sure to write your name, PID, and section on the front of your blue book. This is a closed note, closed book exam. Calculators are not allowed. Show all of your work. No credit will be given for unsupported answers, even if correct. Simplify answers as much as possible.

The second derivative test and general form of the chain rule may be useful at some point on this exam. For your convenience, I have reprinted them here. It is your responsibility to know where and when it makes sense to use this information.

Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y)$.

- If $D(x_0, y_0) > 0$
 - If $f_{xx}(x_0, y_0) > 0$ then (x_0, y_0) is a local minimum.
 - If $f_{xx}(x_0, y_0) < 0$ then (x_0, y_0) is a local maximum.
- If $D(x_0, y_0) < 0$ then (x_0, y_0) is a saddle point.
- If $D(x_0, y_0) = 0$ then the test is inconclusive.

Suppose $f : \mathbb{R}^m \rightarrow \mathbb{R}^p$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and let $\mathbf{D}f$ denote the matrix of partial derivatives of f . If g is differentiable at \mathbf{x}_0 and f is differentiable at $g(\mathbf{x}_0)$ then $f \circ g$ is differentiable at \mathbf{x}_0 and we have:

$$\mathbf{D}(f \circ g)(\mathbf{x}_0) = \mathbf{D}f(g(\mathbf{x}_0))\mathbf{D}g(\mathbf{x}_0).$$

1. (10 points) Let $f(u, v, w) = ve^u \sin(w)$ and suppose that $u(x, y) = y^2$, $v(x, y) = xy$ and $w(x, y) = \frac{\pi x}{y}$. If $h(x, y) = f(u(x, y), v(x, y), w(x, y))$, what is $\frac{\partial h}{\partial y}$ at $(1, 2)$?

Solution: To solve this problem we'll use the chain rule. For this particular problem, the general form of the chain rule tells us that:

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y}.$$

Then we simply compute each partial derivative:

$$\begin{aligned}\frac{\partial f}{\partial u} &= ve^u \sin(w) \\ \frac{\partial u}{\partial y} &= 2y \\ \frac{\partial f}{\partial v} &= e^u \sin(w) \\ \frac{\partial v}{\partial y} &= x \\ \frac{\partial f}{\partial w} &= ve^u \cos(w) \\ \frac{\partial w}{\partial y} &= \frac{-\pi x}{y^2}.\end{aligned}$$

Next we need to evaluate everything at $(1, 2)$. We have $u(1, 2) = 4$, $v(1, 2) = 2$ and $w(1, 2) = \frac{\pi}{2}$. Plugging back into our original equation we have

$$\frac{\partial h}{\partial y} = 2e^4 \sin(\pi/2)(2 \cdot 2) + e^4 \sin(\pi/2)(1) + 2e^4 \cos(\pi/2)(-\pi/2^2) = 9e^4.$$

2. (10 points) A particle moves through space according to the path $c(t) = (t^2, \frac{8}{3}t^{3/2}, 4t)$, which gives the position of the particle after t seconds. What is the distance traveled by the particle during the first 3 seconds?

Solution: We need to compute the arc length of the path as t varies from 0 to 3.

$$c'(t) = (2t, 4\sqrt{t}, 4)$$

so we have

$$\|c'(t)\| = \sqrt{(2t)^2 + (4\sqrt{t})^2 + 4^2} = \sqrt{4(t+2)^2} = 2(t+2) = 2t+4.$$

Thus, the arc length is

$$\int_0^3 2t+4 = t^2 + 4t \Big|_0^3 = 9 + 12 = 21.$$

3. (10 points) Let $f(x, y) = 3y^2 + 2x^2 + 6y$.
- (a) Find all critical points of $f(x, y)$ and classify them. (*local min, local max, or saddle*)

Solution: We begin by computing partial derivatives and setting them equal to 0:

$$\begin{aligned}f_x &= 4x = 0 \\ f_y &= 6y + 6 = 0.\end{aligned}$$

The only critical point is at $(0, -1)$. Now we use the second derivative test to classify. $D(0, -1) = 24 > 0$ and $f_{xx} = 4 > 0$ so the point is a local minimum.

- (b) Use the Lagrange Multipliers method to find all points where local extrema of $f(x, y)$ may occur, subject to the constraint $x^2 + y^2 = 25$. Note: You do not need to evaluate the function at the specified points, or classify them.

Solution: We first find the Lagrange equations by setting $\nabla f = \lambda \nabla g$.

$$(4x, 6y + 6) = \lambda(2x, 2y).$$

By the first Lagrange equation, either $\lambda = 2$ or $x = 0$. If $\lambda = 2$ then by the second Lagrange equation, $y = -3$. Plugging this back into the constraint equation gives $x = \pm 4$. If $x = 0$ then by the constraint equation we have $y = \pm 5$. Finally, we note that $\nabla g = (0, 0)$ only at the origin, which does not lie on the constraint curve, so we need not consider it. Thus, all points where extrema may occur are:

$$(4, -3), (-4, -3), (0, -5), \text{ and } (0, 5).$$

4. (10 points) Let $f(x, y, z) = 2x^3 - 9x^2 + y^2 - 2z$. Find all points where the tangent plane to the level surface defined by $f(x, y, z) = 4$ is parallel to the plane $z = 5y - 6x + 17$.

Solution: Recall that two planes are parallel if their normal vectors are parallel. A normal vector to the given plane $z = 5y - 6x + 17$ is $(-6, 5, -1)$. We also know that the gradient of f points orthogonally to level sets, so a normal vector of the tangent plane to the level surface defined by $f(x, y, z) = 4$ at the point (x, y, z) is given by $\nabla f = (6x^2 - 18x, 2y, -2)$. Thus, to solve this problem we need to find points (x, y, z) which satisfy

$$(6x^2 - 18x, 2y, -2) = \lambda(-6, 5, -1)$$

for some constant λ . By equating the third components we see that $\lambda = 2$. Equating the first and second components then give the equations $6x^2 - 18x = -12$ and $2y = 10$. We can rewrite the first equation as $6(x^2 - 3x + 2) = 6(x - 1)(x - 2) = 0$ so $x = 1$ or 2 . The second equation immediately implies that $y = 5$. Finally, we solve for z by remembering that $f(x, y, z) = 4$. In other words, we need to solve

$$2(1)^3 - 9(1)^2 + 5^2 - 2z = 4$$

and

$$2(2)^3 - 9(2)^2 + 5^2 - 2z = 4.$$

This tells us that the points where the planes are parallel are $(1, 5, 7)$ and $(2, 5, 1/2)$.

5. (10 points) Let $c(t)$ be a path such that $\|c(t)\| = 4$ for all t . Show that $c'(t)$ is orthogonal to $c(t)$. Hint: $\|c(t)\|^2 = c(t) \cdot c(t)$.

Solution: Using the hint we have

$$16 = 4^2 = \|c(t)\|^2 = c(t) \cdot c(t).$$

Differentiating both sides with respect to t and using the dot product rule we have

$$0 = c'(t) \cdot c(t) + c(t) \cdot c'(t) = 2c(t) \cdot c'(t).$$

Therefore $c(t) \cdot c'(t) = 0$ so $c(t)$ and $c'(t)$ are orthogonal.