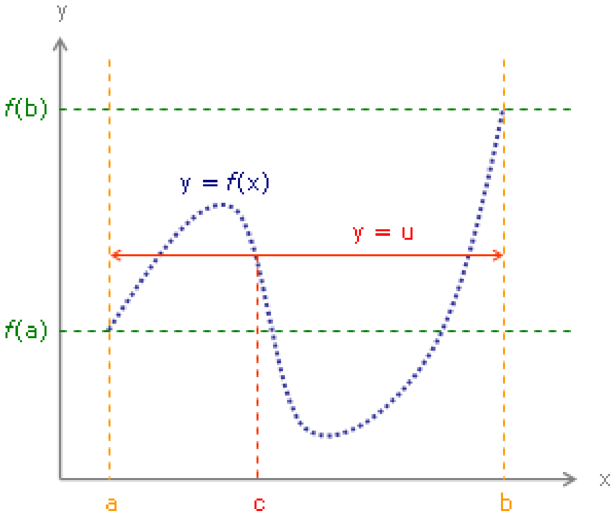
Intermediate Value Theorem

The intermediate value theorem shows one property of continuous functions. The theorem states the following:

Theorem 0.1. *If the function* $y = f(x)$ *is* **continuous** *on the interval* $[a, b]$*, and* u *is a number between* $f(a)$ *and* $f(b)$*, then there exists* $c \in [a, b]$ *such that* $f(c) = u$*.*



This captures an intuitive property of continuous functions: for instance, given f continuous on $[1, 2]$, if $f(1) = 3$ and $f(2) = 5$ then f must take the value 4 somewhere between 1 and 2. It represents the idea that the **graph of a continuous function on a closed interval can be drawn without lifting your pencil from the paper**.

Example. *We want to show that* $\sqrt{2}$ *exists.*

Let's consider a function $f(x) = x^2$ in an interval $[1, 2]$. We know that $f(x)$ is continuous because it is a polynomial, and $f(1) = 1$ and $f(2) = 4$. So, by the intermediate value theorem, for 2 between $f(1) = 1$ and $f(2) = 4$, there exists $c \in (1, 2)$ such that $f(c) = c^2 = 2$. This guarantees the existence of a number c such that $c^2 = 2$ and that is exactly the definition of the number $\sqrt{2}$. Therefore, we showed the existence of $\sqrt{2}$. $\square$

One of the useful consequences of the Intermediate Value Theorem is the following.

Corollary 0.2. *Let* f *be a function which is* **continuous** *on the closed interval* $[a, b]$*. Suppose that the product* $f(a) \cdot f(b) < 0$ *(i.e.,* $f(a)$ *and* $f(b)$ *have different signs); then there exists* c *in* (a, b) *such that* $f(c) = 0$*. In other words,* f **has at least one root in the interval** (a, b)*.*

We can derive this Corollary from the Intermediate Value Theorem, because in the case when $f(a) < 0$ and $f(b) > 0$, $u = 0$ is between $f(a)$ and $f(b)$. So the Intermediate Value Theorem implies the existence of $c \in (a, b)$ such that $f(c) = u = 0$. And this implies that c is a root of $f(x)$.

Example. *Show that the function* $f(x) = \ln(x) - 1$ *has a solution between* 2 *and* 3*.*

If we plug in $x = 2$ and $x = 3$ into $f(x)$, we see that $f(x) = \ln 2 - 1 \approx -0.307 < 0$ and $f(3) = \ln 3 - 1 \approx 0.099 > 0$. So 0 is between $f(2)$ and $f(3)$, and since the function $f(x) = \ln x - 1$ is continuous, by the Intermediate Value Theorem, there is $c \in (2, 3)$ such that $f(c) = \ln c - 1 = 0$. This c is the solution of $f(x)$. ☐

Exercise 1. $f(x)$ is continuous on $(-\infty, \infty)$. Select all statements that are **always** true.

☐ A. $f(x)$ has a root.
☐ B. $g(x) = \sin f(x)$ is continuous.
☐ C. $f(x)$ does not have a minimum.
☐ D. $f(-1) < 0 < f(1)$

2. $g(x)$ is defined on $[-1, 1]$, $f(x) = xg(x)$ is continuous on $[-1, 1]$. Select all statements that are **always** true.

☐ A. $g(x)$ is continuous on [-1,1].
☐ B. $f(x)$ has a maximum on $[-1, 0)$.
☐ C. $f(x)$ has a root on $[-1, 1]$.
☐ D. $g(x)$ has a root on $[-1, 1]$.

3. $f(x)$ is continuous on $[-2, 8)$. Select all statements that are **always** true.

☐ A. $f(x)$ has a maximum on $[-2, 8)$.
☐ B. There exists $c \in (-2, 7)$ such that $f(c) = \dfrac{1}{2}\left(f(-2) + f(7)\right)$.
☐ C. $f(x)$ is not increasing.
☐ D. $\lim\limits_{x \to 8-} \sin f(x)$ exists.

4. $f(x)$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Select all statements that are **always** true.

☐ A. $\lim\limits_{x \to \frac{\pi}{2}+} f(x) = f\left(\dfrac{\pi}{2}\right)$.
☐ B. $f(x)$ has a maximum on $[-\pi/2, \pi/2]$.
☐ C. $g(x) = \dfrac{f(x)}{\cos x}$ is continuous on $[-\pi/2, \pi/2]$.
☐ D. $\lim\limits_{x \to \frac{\pi}{2}-} f(x) = f\left(\dfrac{\pi}{2}\right)$.

Ans : 1. B, 2. C, 3. B, D, 4. B, D