Midterm I for MATH 20A, Fall 2009

1. For each of the following functions $f(x)$, compute its derivative $f'(x)$:

   (a) $f(x) = 2\cos(x)e^x$
   Ans. Apply the *product rule* : $f'(x) = -2\sin(x)e^x + 2\cos(x)e^x$.

   (b) $f(x) = \frac{x^5 - e^x}{5x+1}$
   Ans. Apply the *quotient rule* :

$$f'(x) = \frac{(5x^4 - e^x)(5x+1) - 5(x^5 - e^x)}{(5x+1)^2}$$

   (c) $f(x) = \pi^3(x+1)$
   Ans. $f'(x) = \pi^3$.

2. Find the exact value of

$$\lim_{x\to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$$

   Ans. It's an indeterminate form of type $\frac{0}{0}$. We take care of this case by multiplying the conjugate of the numerator on both of top and bottom.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} &= \lim_{x\to 0} \frac{(\sqrt{2+x} - \sqrt{2})(\sqrt{2+x} + \sqrt{2})}{x(\sqrt{2+x} + \sqrt{2})} \\
&= \lim_{x\to 0} \frac{2+x-2}{x(\sqrt{2+x} + \sqrt{2})} \\
&= \lim_{x\to 0} \frac{x}{x(\sqrt{2+x} + \sqrt{2})} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} \qquad (\leftarrow \text{ now this is continuous around } x = 0) \\
&= \frac{1}{2\sqrt{2}}
\end{aligned}$$

3. Let $f(x) = 5 + \sqrt{x-5}$.
   (a) Determine the domain and range of f.

   Ans. domain $= \{x | x \geq 5\}$, range $= \{y | y \geq 5\}$.

   (b) Find a formula for the inverse $f^{-1}(x)$ and state its domain and range.

   Ans. Note that $f(x)$ is one-to-one on the domain so $f^{-1}(x)$ exists. To find it, we first solve $y = f(x)$ for x, and exchange x and y.

$$\begin{aligned}
y = 5 + \sqrt{x-5} \quad &\Rightarrow \quad \sqrt{x-5} = y - 5 \\
& \quad x - 5 = (y-5)^2 \\
& \quad x = (y-5)^2 + 5 \\
&\Rightarrow \quad y = f^{-1}(x) = (x-5)^2 + 5
\end{aligned}$$

   domain of f^{-1} = range of f $= \{x | x \geq 5\}$, range of f^{-1} = domain of f $= \{y | y \geq 5\}$

4. The line tangent to the curve $y = 4 - x^2$ at the point $(2, 0)$ also passes through the point $(0, a)$. Find a.

Ans. We first find the equation of the tangent line passing at the point $(2, 0)$. Note that the slope of the tangent line is the derivative at that point, i.e.,

$$\text{slope of the tangent line at } (2,0) = f'(2) = -2 \cdot 2 = -4$$

So the tangent line equation with slope -4 passing through $(2, 0)$ is $y - 0 = (-4)(x - 2)$, or

$$y = -4x + 8$$

The fact that the point $(0, a)$ is lying on the line means that $(0, a)$ satisfies the equation $y = -4x + 8$. Therefore,

$$a = -4 \cdot 0 + 8 = 8$$

5. Let

$$g(x) = \begin{cases} x + 1 & \text{if} \quad |x| < 3 \\ b - x^2 & \text{if} \quad |x| \geq 3 \end{cases}$$

(a) Find the constant b so that $g(x)$ is continuous at $x = 3$. Be sure to justify your answer using the definition of continuity.

Ans. To make $g(x)$ be continuous at $x = 3$, it should satisfy

$$\lim_{x \to 3-} g(x) = \lim_{x \to 3+} g(x)$$

$$\Rightarrow 3 + 1 = b - 9$$

$$\Rightarrow b = 13$$

(b) With the choice of b you made above, is $g(x)$ continuous for all real numbers x? Why or why not?

Ans. For $x \in (-\infty, -3) \cup (3, \infty)$, $g(x)$ is defined by $13 - x^2$ which is a polynomial so we know that it is continuous, and for $x \in (-3, 3)$, $g(x)$ is defined by $x + 1$ which is also a polynomial so we know that it is continuous. And we made the function to be continuous at $x = 3$, so the only point left is at $x = -3$. So now we check the continuity at $x = -3$:

$$\lim_{x \to -3-} g(x) = 13 - (-3)^2 = 4, \qquad \lim_{x \to -3+} g(x) = -3 + 1 = -2$$

Since the left-hand limit and the right-hand limit are not equal at $x = -3$, the limit value does not exist, so $g(x)$ is not continuous at $x = -3$.