

Midterm II for MATH 20A, Fall 2009

1. Differentiate the following functions.

$$(a) f(x) = \frac{x}{2 + \sin x}$$

Ans. By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{1 \cdot (2 + \sin x) - x \cdot \cos x}{(2 + \sin x)^2} \\ &= \frac{2 + \sin x - x \cos x}{(2 + \sin x)^2} \end{aligned}$$

$$(b) g(x) = e^{3x} \cos(4x)$$

Ans. By the product rule,

$$g'(x) = 3e^{3x} \cos(4x) - 4e^{3x} \sin(4x)$$

$$(c) h(x) = [\cos x]^x$$

Ans. We do the logarithmic differentiation ; first, take the natural log on both sides of the equation

$$\ln h(x) = \ln([\cos x]^x) = x \ln(\cos x).$$

Differentiate by using the chain rule :

$$\frac{h'(x)}{h(x)} = \ln(\cos x) + x \cdot \frac{-\sin x}{\cos x} = \ln(\cos x) - x \tan x$$

so,

$$\begin{aligned} h'(x) &= h(x) \cdot (\ln(\cos x) - x \tan x) \\ &= (\cos x)^x (\ln(\cos x) - x \tan x). \end{aligned}$$

2. Let $f(x) = x + \frac{1}{x-3}$. The first and second derivatives of f are given by

$$f'(x) = 1 - \frac{1}{(x-3)^2} \quad \text{and} \quad f''(x) = \frac{2}{(x-3)^3}.$$

(a) Find the interval(s) on which f is increasing and the interval(s) on which f is decreasing.

Ans. Note that

$$\begin{aligned} f' > 0 &\Leftrightarrow f \text{ is increasing} \\ f' < 0 &\Leftrightarrow f \text{ is decreasing} \end{aligned}$$

To find the interval on which f is increasing, we solve the inequality $f' > 0$:

$$f'(x) = \frac{(x-3)^2 - 1}{(x-3)^2} > 0$$

Since the denominator is always positive, $f' > 0$ if $(x - 3)^2 - 1 > 0$.

$$(x - 3)^2 - 1 = x^2 - 6x + 8 = (x - 2)(x - 4) > 0 \text{ if } x < 2 \text{ or } x > 4.$$

So, f is increasing on the interval $(-\infty, 2) \cup (4, \infty)$.

And, $f' < 0$ if $2 < x < 4$, from the inequality $(x - 2)(x - 4) < 0$. So we get that f is decreasing on the interval $(2, 4)$.

(b) Find the local maximum and local minimum values of f .

Ans. We find the critical values first, by solving $f'(x) = 0$.

$$f'(x) = 0 \text{ if } (x - 3)^2 - 1 = (x - 2)(x - 4) = 0$$

So we get the critical values $x = 2, 4$.

Note *the second derivative test* that at a critical value $x = c$, if $f''(c) < 0$ then $f(c)$ is a local maximum; and if $f''(c) > 0$ then $f(c)$ is a local minimum.

$$f''(2) = -2 < 0 \Rightarrow f(2) = 1 \text{ is a local maximum}$$

$$f''(4) = 2 > 0 \Rightarrow f(4) = 5 \text{ is a local minimum.}$$

(c) Find the interval(s) on which the graph of f is concave upward and the interval(s) on which the graph f is concave downward.

Ans. Note that we get the information about the concavity of the graph by the sign of the second derivative ;

$$f''(x) > 0 \Leftrightarrow \text{concave upward}$$

$$f''(x) < 0 \Leftrightarrow \text{concave downward}$$

Note that the sign of $f''(x) = \frac{2}{(x-3)^3}$ is determined by the sign of $(x - 3)^3$.

$f''(x) > 0$ if $(x - 3)^3 > 0 \Leftrightarrow x > 3$; and $f''(x) < 0$ if $(x - 3)^3 < 0 \Leftrightarrow x < 3$. Thus,

$$(3, \infty); \text{ concave upward}$$

$$(-\infty, 3); \text{ concave downward}$$

(d) Find the vertical asymptote(s), if any.

Ans. $f(x) = x + \frac{1}{x-3} = \frac{x^2-3x+1}{x-3}$ has a vertical asymptote when the denominator is zero. So the vertical asymptote is $x = 3$.

3. Use the fact that $27^{1/3} = 3$ to find a linear approximation for $(27.03)^{1/3}$.

Ans. Let $f(x) = x^{1/3}$ and use the linearization of $f(x)$ at $a = 27$. Since $f'(x) = \frac{1}{3}x^{-2/3}$, the linearization of $f(x)$ at $a = 27$ is

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 3 + \frac{1}{3}(27)^{-2/3}(x - 27) \\ &= 3 + \frac{1}{27}(x - 27). \end{aligned}$$

$$(27.03)^{1/3} = f(27.03) \approx L(27.03) = 3 + \frac{1}{27}(27.03 - 27) = 3 + \frac{1}{27}(0.03) = \frac{2701}{900}$$

4. Randall Cohn has a pool with the shape of an inverted cone which is 5 meters deep with a radius of 5 meters at the top (base). Randall fills the pool with his garden hose at a rate of 0.1 cubic meters per minute. At what rate is the water depth increasing when the depth is 3 meters? (Note : the volume of a cone of height h and radius r is given by

$$V = \frac{1}{3}\pi r^2 h. \quad)$$

Ans. Given $\frac{dV}{dt} = 0.1 \text{ m}^3/\text{min}$, we want to find $\frac{dh}{dt}$ when $h = 3$. We find an equation which relates V and h first. To get rid of r in the expression of V , we use the ratio of the radius and the surface of the water and the height of the water :

$$\frac{5}{5} = \frac{r}{h} \Rightarrow r = h$$

So,

$$V = \frac{1}{3}\pi h^3$$

Differentiate with respect to t by using the chain rule :

$$\begin{aligned} \frac{dV}{dt} &= \pi h^2 \frac{dh}{dt} \\ \Rightarrow \frac{dh}{dt} &= \frac{1}{\pi h^2} \frac{dV}{dt} = \frac{0.1}{\pi h^2} \end{aligned}$$

When $h = 3$;

$$\left. \frac{dh}{dt} \right|_{h=3} = \frac{1}{90\pi}$$

5. Find the point(s) on the ellipse $x^2 + xy + y^2 = 12$ at which the corresponding tangent line is horizontal.

Ans. Noting that the tangent line is horizontal when $y' = 0$, we first find y' by the implicit differentiation ;

$$\begin{aligned} \frac{d}{dx}(x^2 + xy + y^2) &= \frac{d}{dx}(12) \\ 2x + y + xy' + 2yy' &= 0 \\ (x + 2y)y' &= -(2x + y) \end{aligned}$$

so,

$$y' = -\frac{2x + y}{x + 2y}$$

$y' = 0$ if $2x + y = 0$ i.e., $y = -2x$. To find the points on the curve, we use $y = -2x$ for y in the equation :

$$\begin{aligned} x^2 + x(-2x) + (-2x)^2 &= 12 \\ x^2 - 2x^2 + 4x^2 &= 12 \\ 3x^2 &= 12 \\ x^2 &= 4 \end{aligned}$$

So $x = 2, -2$ and use $y = -2x$ to find the corresponding y values. Hence, the tangent line is horizontal at $(2, -4)$ and $(-2, 4)$.