Solution of Quiz 3 for MATH 20C, 2009

1. Describe the domain and the range of the function

$$f(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$$

Ans.

$$\begin{aligned} \text{Domain of } f &= \{(x,y,z) | 9 - x^2 - y^2 - z^2 \geq 0\} \\ &= \{(x,y,z) | x^2 + y^2 + z^2 \leq 9\} \end{aligned}$$

which is the interior and the boundary of a sphere centered at the origin with radius 3.

For the range of f, if we let $w = x^2 + y^2 + z^2$, by the domain of f, $0 \leq w \leq 9$, so $0 \leq \sqrt{9 - w} \leq 3$. Hence,

$$\text{Range of } f = \{w | 0 \leq w \leq 3\}.$$

2. Find an arc length parametrization of the circle in the plane $z = 9$ with radius 4 and center $(1, 4, 9)$. Note that the *arc length function* is defined by

$$s(t) = \int_0^t ||r'(u)|| du.$$

Ans. A parametrization of a circle with radius 4 centered at $(1, 4, 9)$ on $z = 9$ is

$$r(t) = < 4\cos t + 1, 4\sin t + 4, 9 > .$$

$$r'(t) = < -4\sin t, 4\cos t, 0 > \quad \Rightarrow \quad ||r'(t)|| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4.$$

So,

$$s = \int_0^t 4du = 4t \quad \Rightarrow t = \frac{s}{4}.$$

Hence, the arc length parametrization is

$$r_1(s) = r(s/4) = \left\langle 4\cos\left(\frac{s}{4}\right) + 1, 4\sin\left(\frac{s}{4}\right) + 4, 9 \right\rangle.$$

3. Find $\mathbf{r}(t)$ and $\mathbf{v}(t)$ given $\mathbf{a}(t)$ and the initial velocity and position

$$\mathbf{a}(t) = t\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i}, \quad \mathbf{r}(0) = \mathbf{j}.$$

Ans.

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(s)ds + \mathbf{v}(0) = \left(\int_0^t s ds\right)\mathbf{k} + \mathbf{i} = \mathbf{i} + \frac{1}{2}t^2\mathbf{k} = \left\langle 1, 0, \frac{1}{2}t^2 \right\rangle.$$

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(s)ds + \mathbf{r}(0) = \left[\left(\int_0^t 1ds\right)\mathbf{i} + \left(\int_0^t \frac{1}{2}s^2 ds\right)\mathbf{k}\right] + \mathbf{j} = t\mathbf{i} + \mathbf{j} + \frac{1}{6}t^3\mathbf{k} = \left\langle t, 1, \frac{1}{6}t^3 \right\rangle.$$