Quiz 7 Solution for MATH 20C, 2009

1. For the following double integral, (a) sketch the region and (b) change the order of integration and evaluate.

$$\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$$

Ans. (b) The domain of integration is

$$D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

$$\begin{aligned} \int_0^1 \int_0^x \frac{\sin x}{x} dy dx &= \int_0^1 \left[\frac{\sin x}{x} \cdot y \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \sin x dx \\ &= [-\cos x]_0^1 \\ &= -\cos(1) + 1 \end{aligned}$$

2. Find the volume of the solid in $\mathbf{R}^3$ bounded by $y = x^2$, $x = y^2$, $z = x + y + 5$, and $z = 0$.

Ans. The solid is the set of points $\{(x, y, z) | 0 \leq z \leq x + y + 5, x^2 \leq y \leq \sqrt{x}, 0 \leq x \leq 1\}$. So,

$$\begin{aligned} \text{Volume} &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y+5} 1 dz dy dx \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} x + y + 5 dy dx \\ &= \int_0^1 \left[xy + \frac{1}{2} y^2 + 5y \right]_{y=x^2}^{y=\sqrt{x}} dx \\ &= \int_0^1 x\sqrt{x} + \frac{1}{2}x + 5\sqrt{x} - x^3 - \frac{1}{2}x^4 - 5x^2 dx \\ &= \left[\frac{2}{5}x^2\sqrt{x} + \frac{1}{4}x^2 + \frac{10}{3}x\sqrt{x} - \frac{1}{4}x^4 - \frac{1}{10}x^5 - \frac{5}{3}x^3 \right]_0^1 \\ &= \frac{2}{5} + \frac{1}{4} + \frac{10}{3} - \frac{1}{4} - \frac{1}{10} - \frac{5}{3} \\ &= \frac{59}{30}. \end{aligned}$$

3. Let $\mathcal{W}$ be the region above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$. Write down the triple integral for the volume of $\mathcal{W}$ in cylindrical coordinates. (Do not evaluate).

Ans. The intersection of two surfaces is $x^2 + y^2 = 2$ when $z = 2$. So, the projected down image of $\mathcal{W}$ on the xy-plane is the disk $\{(x, y) | x^2 + y^2 \leq 2\}$.

$$\text{Volume of } \mathcal{W} = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\sqrt{6-r^2}}^{4-r^2} 1 dz r dr d\theta.$$