Review with Solution for Midterm I, MATH 20C, 2009

1 Chapter 12.

1. [12.2] Find an equation of a sphere if one of its diameters has end-points $(2, 1, 4)$ and $(4, 3, 10)$.

Ans. The center of the sphere is the mid-point of the given two end points of one diameter, i.e.,

$$\text{center} = \left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = (3, 2, 7).$$

The radius of the sphere is the distance between either one of the end points of the given diameter and the center of the sphere ;

$$\text{radius} = \sqrt{(3-2)^2 + (2-1)^2 + (7-4)^2} = \sqrt{11}.$$

Therefore, the sphere equation is

$$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11. \qquad \square$$

2. [12.2, 12.3, 12.4] Given $P = (2, 1, 3)$, $Q = (1, -1, 0)$, $R = (1, 1, -1)$, $S = (0, 1, -2)$ determine whether the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$
(a) are parallel
(b) are perpendicular
by using the cross and dot products.

Ans. (a) $\overrightarrow{PQ} =< -1, -2, -3 >$, $\overrightarrow{RS} =< -1, 0, -1 >$. Note that two vectors are parallel if and only if the cross product of them is zero.

$$\overrightarrow{PQ} \times \overrightarrow{RS} = \begin{vmatrix} i & j & k \\ -1 & -2 & -3 \\ -1 & 0 & -1 \end{vmatrix} = 2i - (-2)j - 2k = 2i + 2j - 2k \neq 0.$$

Hence, they are **not** parallel.
(b) Also, note that two vectors are perpendicular if and only if the dot product of two vectors is zero.

$$\overrightarrow{PQ} \bullet \overrightarrow{RS} = (-1)\cdot(-1) + (-2)\cdot 0 + (-3)\cdot(-1) = 1 + 3 = 4 \neq 0.$$

Hence, they are **not** perpendicular. $\square$

3. [12.3, 12.4] Let $\mathbf{a} =< 2, 2, 2 >$ and $\mathbf{b} =< 5, c, 5 >$ be vectors, where c is to be determined.
(a) For what value of c is $\mathbf{b} =< 5, c, 5 >$ orthogonal to $\mathbf{a} =< 2, 2, 2 >$?
(b) For what value of c is $\mathbf{b} =< 5, c, 5 >$ parallel to $\mathbf{a} =< 2, 2, 2 >$?

Ans. (a) $\mathbf{a} \bullet \mathbf{b} = 10 + 2c + 10 = 0$, if $c = -10$. So they are perpendicular when $c = -10$.
(b) Parallel means $\mathbf{a} = \lambda \mathbf{b}$ for some constant λ. This means that we must have $2 = 5\lambda$, $2 = c\lambda$. From the first equation, we get $\lambda = 2/5$ and plugging this in to the second gives $c = 5$. $\square$

4. [12.4] Let $A = (2, 1, -1)$, $B = (3, 0, -2)$, $C = (3, 2, 1)$ and $D = (-2, 0, 1)$.
(a) Find the area of the parallelogram that has AB and AC as adjacent sides.
(b) Find the volume of the parallelepiped that has edges AB, AC, and AD.

Ans. (a) Let $u =< 1, -1, -1 >$ be the vector from A to B and let $v =< 1, 1, 2 >$ be the vector from A to C. Then

$$\text{Area} = ||u \times v|| = || < -1, -3, 2 > || = \sqrt{1 + 9 + 4} = \sqrt{14}.$$

(b) Let $w =< -4, -1, 2 >$ be the vector from A to D. Then

$$\text{Volume} = |(u \times v) \bullet w| = | < -1, -3, 2 > \bullet < -4, -1, 2 > | = |4 + 3 + 4| = 11$$

where u and v are the same as in the solution to part (a). □

5. Let $\mathbf{a} =< 1, 1, 1 >$ and $\mathbf{b} =< -2, -2, 1 >$.
(a) [12.4] Find two vectors that are orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
(b) [12.4] Find the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$.

Ans. (a) First of all, we know that the cross product of $\mathbf{a}$ and $\mathbf{b}$ is orthogonal to both of them. So, one of two vectors that are orthogonal to them is

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -2 & -2 & 1 \end{vmatrix} = 3i - 3j =< 3, -3, 0 > .$$

The other one could be any vector which is a scalar multiple of $\mathbf{v}$, i.e., $\mathbf{w} = \mathbf{v}/3 =< 1, -1, 0 >$.
(b) Note that $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \sin\theta$. So,

$$|| < 3, -3, 0 > || = || < 1, 1, 1 > || \cdot || < -2, -2, 1 > || \sin\theta$$

$$\Rightarrow \sin\theta = \frac{\sqrt{9+9}}{\sqrt{1+1+1} \cdot \sqrt{4+4+1}} = \frac{3\sqrt{2}}{3\sqrt{3}} = \sqrt{\frac{2}{3}}. \qquad \square$$

6. [12.4, 12.5] Given the points $P = (2, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, -5)$.
(a) Find a vector that is normal to the plane through P, Q and R.
(b) At which point does the line $x = 1 + t$, $y = 1 - t$, $z = -3 - 3t$ intersect the plane?

Ans. (a) $\overrightarrow{PQ} =< -2, 1, 0 >$, $\overrightarrow{PR} =< -2, 0, -5 >$. $\overrightarrow{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = -5i - 10j + 2k$.
(b) To find the plane equation, we just pick one of the given points, e.g., $P = (2, 0, 0)$ and get $-5(x-2) - 10y + 2z = 0$, or, $5x + 10y - 2z = 10$. If the point of the given line is one the plane, then the point $(1 + t, 1 - t, -3 - 3t)$ should satisfy the plane equation.

$$5(1 + t) + 10(1 - t) - 2(-3 - 3t) = 10$$

$$\Rightarrow t = -11$$

Therefore, the line intersects the plane at the point $(-10, 12, 30)$ when $t = -11$. □

7. [12.4, 12.5] Let $P(1, 0, -3)$, $Q(0, -2, -4)$, and $R(4, 1, 6)$ be points.
(a) Find the equation of the plane through the points P, Q and R.
(b) Find the area of the triangle with vertices P, Q and R.

Ans. Let $\mathbf{a} = \overrightarrow{PQ} =< -1, -2, -1 >$ and $\mathbf{b} = \overrightarrow{PR} =< 3, 1, 9 >$. A normal vector to the plane is

$$n = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ -1 & -2 & -1 \\ 3 & 1 & 9 \end{vmatrix} = (-18 + 1)i - (-9 + 3)j + (-1 + 6)k = -17i + 6j + 5k.$$

Since P is in the plane, the plane equation is $-17(x - 1) + 6y + 5(z + 3) = 0$.
(b) The area is $A = ||\mathbf{a} \times \mathbf{b}||/2 = \sqrt{17^2 + 6^2 + 5^2}/2$. □

8. [12.4, 12.5] (a) Find a unit vector which is orthogonal to $< 1, -1, -1 >$ and $< 1, 1, 2 >$.
(b) Find the line of intersection of the two planes

$$x - y - z = 2, \quad x + y + 2z = -2.$$

Ans. (a) Note that the result of the cross product of two vectors is orthogonal to both of the two vectors. So, we first to the cross product with two given vectors and get $< 1, -1, -1 > \times < 1, 1, 2 > = < -1, -3, 2 >$. To create a unit vector, we simply divide $< -1, -3, 2 >$ by it's length which is $|| < -1, -3, 2 > || = \sqrt{1+9+4} = \sqrt{14}$. Therefore, $\frac{1}{\sqrt{14}} < -1, -3, 2 >$ is a unit vector orthogonal to both $< 1, -1, -1 >$ and $< 1, 1, 2 >$.

(b) Note that the normal vectors of the two planes are the vectors given in part (a). If we sole two linear equations

$$\begin{cases} x - y - z &= 2 \\ x + y + 2z &= -2 \end{cases}$$

we get the solution $(t, 3t - 2, -2t)$ for any $t \in \mathbb{R}$. Therefore, the parametric equations of the line of intersection are

$$\begin{cases} x(t) &= t, \\ y(t) &= 3t - 2, \\ z(t) &= -2t \end{cases}$$

9. [12.5] Find an equation for the plane containing the lines

$$\lambda(t) = < 1, 3, 5 > + t < 1, 4, 7 > \quad \text{and} \quad \mu(t) = < 1, 3, 5 > + t < 2, -1, 4 > .$$

Ans. The normal vector of the plane containing the two given lines is orthogonal to both of the direction vectors of the lines, i.e.,

$$\overrightarrow{n} = < 1, 4, 7 > \times < 2, -1, 4 > = < 23, 10, -9 > .$$

We need to have one point of the plane to get the plane equation, and we can observe that $(1, 3, 5)$ is on both of the given lines and hence on the plane. So, the plane equation is

$$23(x - 1) + 10(y - 3) - 9(z - 5) = 0. \qquad \square$$

10. [12.5] (a) Find the equation of the plane which goes through the points $(1, 1, -1)$, $(0, 1, 0)$ and $(1, -1, 0)$.
(b) Compute the perpendicular distance from the point $(1, 1, 1)$ to the plane described in part (a).

Ans. (a) Let $u = < -1, 0, 1 >$ be the vector that goes from $(1, 1, -1)$ to $(0, 1, 0)$. Let $v = < 1, -2, 0 >$ be the vector that goes from $(0, 1, 0)$ to $(1, -1, 0)$. Since $u \times v = < 2, 1, 2 >$ is orthogonal to the plane, the equation of the plane is of the form

$$2x + y + 2z = d.$$

To solve for d, plug in any one of the three points given. Using the point $(0, 1, 0)$, we see that $d = 1$.

(b) Let w be a vector that goes from any point on the plane to the point $(1, 1, 1)$. Using the point $(1, 1 - 1)$, $w = < 0, 0, 2 >$. Therefore the perpendicular distance is given by

$$\frac{|(u \times v) \bullet w|}{|u \times v|} = \frac{4}{\sqrt{4+1+4}} = \frac{4}{3}$$

where u and v are the same as in the solution to part (a). $\square$

11. [12.5] (a) Show that the four points $(2, 0, -3)$, $(0, 5, 4)$, $(1, 1, -1)$ and $(5, -12, -18)$ lie in a plane.
(b) Find the distance between this plane and the parallel plane $x - y + z = 2$.

(*Hint: Find the distance between a point on one plane and the other plane.*)

Ans. (a) To show that the four points are in the same plane, we first find a plane equation containing three of them, and we'll show that the fourth point also satisfies the plane equation. We let $P = (2, 0, -3)$, $Q = (0, 5, 4)$, $R = (1, 1, -1)$, $S = (5, -12, -18)$. Then we find the plane containing P, Q and R, first. The normal vector is

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} =< -2, 5, 7 > \times < -1, 1, 2 >=< 3, -3, 3 > .$$

By using the point $P(2, 0, -3)$, the plane equation is $3(x - 2) - 3y + 3(z + 3) = 0$, or $x - y + z = -1$. If we plug in $x = 5, y = -12, z = -18$ in this plane equation, the equation is true which means the point S satisfies the plane equation as well. Hence, all four points lie in one plane.
(b) We simply find the distance between P and the plane $x - y + z - 2 = 0$.

$$d = \frac{|2 - 0 - 3 - 2|}{\sqrt{1 + 1 + 1}} = \frac{3}{\sqrt{3}} = \sqrt{3}. \qquad \square$$

2 Chapter 13.

1. A particle's position functions is $\mathbf{r}(t) =< 2\cos t, 2\sin t, 2t >$ for $0 \le t \le 2\pi$.
(a) [13.1, 13.2] Find the particle's velocity $\mathbf{v}(t)$ and speed $|\mathbf{v}(t)|$ as a function of time t.
(b) [13.2] Find the particle's acceleration $\mathbf{a}(t)$ as a functions of time t.
(c) [12.3, 13.2] Find the angle between the particle's position $\mathbf{r}(t)$ and acceleration $\mathbf{a}(t)$ as a function of time t.
(d) [13.3] Determine how far the particle traveled during the time interval $0 \le t \le 2\pi$.

Ans. (a) $\mathbf{v}(t) = r'(t) =< -2\sin t, 2\cos t, 2 >$, $||\mathbf{v}(t)|| = \sqrt{4\sin^2 t + 4\cos^2 t + 4} = 2\sqrt{2}$.
(b) $\mathbf{a}(t) = \mathbf{v}'(t) =< -2\cos t, -2\sin t, 0 >$.
(c) $\mathbf{r}(t) \bullet \mathbf{a}(t) = ||\mathbf{r}(t)|| \cdot ||\mathbf{a}(t)|| \cos\theta \Rightarrow -4\cos^2 t - 4\sin^2 t = \sqrt{4\cos^2 t + 4\sin^2 t + 4t} \cdot \sqrt{4\cos^2 t + 4\sin^2 t}\cos\theta$
So,

$$\theta = \cos^{-1}\left(\frac{-4}{4\sqrt{1+t^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{1+t^2}}\right).$$

(d)

$$\text{Length} = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt = \int_0^{2\pi} 2\sqrt{2} dt = 4\sqrt{2}\pi. \qquad \square$$

2. [13.1, 13.2] Consider the curve defined parametrically by the equations

$$x(\theta) = 2\theta - \pi\sin\theta, \quad y(\theta) = 2 - \pi\cos\theta$$

for $-\pi \le \theta \le \pi$.
(a) Find the point where the curve intersects itself.
(b) Write down the equation for each line tangent to the curve at the point you found in part (a).

Ans. (a) Using the table below, we see that the curve intersects itself at the point $(0, 2)$, corresponding to $\theta = \pm\pi/2$.

θ	$x(\theta)$	$y(\theta)$
$-\pi$	-2π	$2+\pi$
$-\pi/2$	0	2
0	0	$2-\pi$
$\pi/2$	0	2
π	2π	$2+\pi$

(b) The slope of the tangent line is given by $y'(\theta)/x'(\theta)$ where $x'(\theta) = 2 - \pi\cos\theta$ and $y'(\theta) = \pi\sin\theta$. For $\theta = \pi/2$, the slope of the tangent line is $\pi/2$ and the equation of the tangent line is $y = \frac{\pi}{2}x + 2$. For $\theta = -\pi/2$, the slope of the tangent line is $-\pi/2$ and the equation of the tangent line is $y = -\frac{\pi}{2}x + 2$. □

3. [13.1, 13.2] Consider the curve defined by the parametric equations

$$x = t, \quad y = 2t^{3/2}, \quad \text{where} \quad 0 \le t \le 2.$$

(a) Write down the equation for the line that is tangent to the curve when $t = 1$.
(b) Find the arc length of the curve.

Ans. (a) $dy/dx = y'(t)/x'(t) = 3t^{1/2} = 3$ and $(x, y) = (1, 2)$, when $t = 1$, so the equation of the tangent line is

$$y - 2 = 3(x - 1).$$

(b)

$$L = \int_0^1 \sqrt{1+9t}dt = \frac{2}{27}(1+9t)^{3/2}\Big|_0^2 = \frac{2}{27}(19^{3/2} - 1). \quad \square$$

4. [13.2] Consider the curve C given by the parametric equations

$$x = \cos t, \quad t = 2 + \sin t, \quad \text{where} \quad 0 \le t \le 2\pi.$$

Find the points where the tangent to the curve is vertical.

Ans. $x'(t) = -\sin t = 0$ when $t = 0$ or $t = \pi$. □

5. [13.2] The curve is given by $x = \sin t$, $y = 1 + \sin t \cos t$ has two tangents at the point $(0, 1)$.
(a) Find the tangents.
(b) Find the angle between them.

Ans. If we let $r(t) =< \sin t, 1 + \sin t \cos t >$, then $r'(t) =< \cos t, \cos^2 t - \sin^2 t >$. The point $(0, 1)$ is obtained when $t = 0$, or $t = \pi$. When $t = 0$, $r'(0) =< 1, 1 >$, and when $t = \pi$, $r'(\pi) =< -1, 1 >$. Hence, $< 1, 1 >$ and $< -1, 1 >$ are the two tangent vectors are $(0, 1)$.
(b) By using the dot product of those two vectors, $< 1, 1 > \bullet < -1, 1 >= || < 1, 1 > || \cdot || < -1, 1 > || \cos\theta$, and that gives $\cos\theta = 0$, so $\theta = \pi/2$. □

6. [13.2] Consider the curve C given by $x = \frac{1}{2}\cos 2t$, $y = 1 + \sin t \cos t$, $t \in \left[0, \frac{\pi}{2}\right]$.
Determine if the tangent of C is horizontal at any point (and find the point).

Ans. Note that the tangent of C is horizontal when $dy/dt = 0$ and $dx/dt \ne 0$.
$dy/dt = \cos^2 t - \sin^2 t = \cos 2t = 0 \Rightarrow 2t = \pi/2, 3\pi/2 \Rightarrow t = \pi/4, 3\pi/4$. (Note that $dx/dt = -\sin 2t$ is not zero when $t = \pi/4$ or $t = 3\pi/4$). Therefore, the tangent of C is horizontal at $(0, 3/2)$ when $t = \pi/4$, and at $(0, 1/2)$ when $t = 3\pi/4$.
□

7. Let $r(t) =< 3t + 1, 4t - 5, 2t >$.
(a) Calculate $s(t) = \int_0^t ||r'(u)||du$ as a function of t.

(b) Find the inverse $\varphi(s) = t(s)$ and show that $r_1(s) = r(\varphi(s))$ is an arc length parametrization.

Ans. (a) $r'(t) =< 3, 4, 2 >$, and so $||r'(t)|| = \sqrt{29}$.

$$s(t) = \int_0^t ||r'(u)||du = \int_0^t \sqrt{29}du = \sqrt{29}t.$$

(b) $t = \varphi(s) = \frac{s}{\sqrt{29}}$

$$r_1(s) = r(\varphi(s)) = r\left(\frac{s}{\sqrt{29}}\right) = \left\langle \frac{3}{\sqrt{29}}s + 1, \frac{4}{\sqrt{29}}s - 5, \frac{2}{\sqrt{29}}s \right\rangle$$

$$r_1'(s) = \left\langle \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{2}{\sqrt{29}} \right\rangle \quad \Rightarrow \quad ||r_1'(s)|| = \sqrt{\frac{9 + 16 + 4}{29}} = 1$$