

Review Solution for Midterm II, MATH 20C, 2009

1 Chapter 13

1. A parking ramp follows the curve in space traced by the function

$$r(t) = \langle \cos t, \sin t, t \rangle \quad \text{for } 0 < t < 4\pi.$$

What is the curvature function $\kappa(t)$ of the ramp?

Ans. Note that the curvature formula is

$$\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}.$$

$r'(t) = \langle -\sin t, \cos t, 1 \rangle$, $r''(t) = \langle -\cos t, -\sin t, 0 \rangle$, $r'(t) \times r''(t) = \langle \sin t, -\cos t, 1 \rangle$. So, $\|r'(t) \times r''(t)\| = \sqrt{2}$. $\|r'(t)\| = \sqrt{2}$. Hence, $\kappa(t) = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}$.

2. (13.5) Santa Claus' sleigh loses power after taking off at 64 ft/s from a 32 foot high roof at an angle 45° . How many seconds elapse before he hits the ground?

(For this problem, assume that Santa exists and that his magic is not powerful enough to alter the earth's gravitational acceleration of 32 ft/s^2).

Ans. $a(t) = \langle 0, -32 \rangle$, $v(0) = 64 \langle \cos 45^\circ, \sin 45^\circ \rangle$, $r(0) = (0, 32)$.

$$v(t) = \int_0^t a(u) du + v(0) = \langle 32\sqrt{2}, -32t + 32\sqrt{2} \rangle,$$

$$r(t) = \int_0^t v(u) du + r(0) = \langle 32\sqrt{2}t, -16t^2 + 32\sqrt{2}t + 32 \rangle.$$

We want to find t such that $-16t^2 + 32\sqrt{2}t + 32 = 0$, or $t^2 - 2\sqrt{2}t - 2 = 0$ and the positive root of this equation is $t = 2 + \sqrt{2}$.

3. (13.5) At $t = 0$, an airplane takes off. At that moment, its position vector is $\langle 0, 0, 0 \rangle$ and its velocity vector is $\langle 1, 2, 0 \rangle$. Find its position vector at time $t = 6$, if the acceleration of the airplane is $a(t) = \langle 1, 0, t \rangle$.

Ans. $r(0) = \langle 0, 0, 0 \rangle$, $v(0) = \langle 1, 2, 0 \rangle$, $a(t) = \langle 1, 0, t \rangle$.

$$v(t) = \int_0^t a(u) du + v(0) = \left\langle t, 0, \frac{1}{2}t^2 \right\rangle + \langle 1, 2, 0 \rangle = \left\langle t + 1, 2, \frac{1}{2}t^2 \right\rangle,$$

$$r(t) = \int_0^t v(u) du + r(0) = \left\langle \frac{1}{2}t^2 + t, 2t, \frac{1}{6}t^3 \right\rangle.$$

Hence, the position at $t = 6$ is $r(6) = \langle 24, 12, 36 \rangle$.

2 Chapter 14

1. (14.2) Compute the following limits. If the limit does not exist, explain why.

$$(a) \quad \lim_{(x,y) \rightarrow (2,3)} \frac{x^2 + xy + 2y^2 - 1}{x^2 - y^2 + 4}$$

$$(b) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 + xy + y^2}$$

Ans. (a) For this limit, we can simply plug in the values $x = 2$ and $y = 3$ because the rational function $\frac{x^2 + xy + 2y^2 - 1}{x^2 - y^2 + 4}$ is defined and continuous at $(2, 3)$. Hence,

$$\lim_{(x,y) \rightarrow (2,3)} \frac{x^2 + xy + 2y^2 - 1}{x^2 - y^2 + 4} = \frac{4 + 6 + 18 - 1}{4 - 9 + 4} = -27.$$

(b) In this case, the limit does not exist because we get two different values if we take the limit along two different paths through the point $(0, 0)$. For example, if we let $y = 0$ then along the x -axis, the limit becomes

$$\lim_{(x,0) \rightarrow (0,0)} \frac{x^2 + 0^2}{x^2 + x \cdot 0 + 0^2} = \lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2} = 1.$$

On the other hand, if we let $y = x$, then the limit becomes

$$\lim_{(x,x) \rightarrow (0,0)} \frac{x^2 + x^2}{x^2 + x \cdot x + x^2} = \lim_{(x,x) \rightarrow (0,0)} \frac{2x^2}{3x^2} = \frac{2}{3}.$$

2. (14.3) Let $f(x, y, z) = e^{xy^2} + \ln(y + z^3)$. Compute the following partial derivatives of f .

(a) f_x

(b) f_y

(c) f_{zx}

(d) f_{zy}

Ans. (a) $f_x = y^2 e^{xy^2}$.

(b) $f_y = 2xy e^{xy^2} + \frac{1}{y+z^3}$.

(c) $f_{zx} = f_{xz} = \frac{\partial}{\partial z} f_x = \frac{\partial}{\partial z} (y^2 e^{xy^2}) = 0$.

(d) $f_{zy} = f_{yz} = \frac{\partial}{\partial z} f_y = \frac{\partial}{\partial z} \left(2xy e^{xy^2} + \frac{1}{y+z^3} \right) = \frac{-3z^2}{(y+z^3)^2}$.

3. (14.4) Let $F(x, y, z) = z - y^2 + x^2 + 3$. Find the tangent plane to the level surface $F(x, y, z) = 0$ at the point $(1, 2, 0)$.

Ans. The tangent plane equation at $(1, 2, 0)$ is

$$F_x(1, 2, 0)(x - 1) + F_y(1, 2, 0)(y - 2) + F_z(1, 2, 0)(z - 0) = 0.$$

Since $\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, -2y, 1 \rangle$, we have $\nabla F(1, 2, 0) = \langle 2, 4, -1 \rangle$ so the equation of the tangent plane is

$$2(x - 1) - 4(y - 2) + (z - 0) = 0$$

or,

$$2x - 4y + z = -6.$$

4. (14.4) Consider the surface given by $z = f(x, y) = \sqrt{x^2 + 3y^2}$.

(a) Find the tangent plane to the surface at the point $(1, 1, 2)$.

(b) A student was asked to find an approximation for $f(1.1, 1.2)$ but the professor did not allow calculators. The student noticed that $f(1.1, 1.2)$ is approximately $f(1, 1) = \sqrt{1 + 3} = 2$. Use the linear approximation to get a better approximation.

Ans. (a)

$$f_x = \frac{x}{\sqrt{x^2 + 3y^2}}, \quad f_y = \frac{3y}{\sqrt{x^2 + 3y^2}} \quad \Rightarrow \quad \nabla f(1, 1) = \langle f_x(1, 1), f_y(1, 1) \rangle = \frac{1}{2} \langle 1, 3 \rangle .$$

The tangent plane equation is

$$z - 2 = \frac{1}{2}(x - 1) + \frac{3}{2}(y - 1) \quad \text{or,} \quad z = \frac{1}{2}x + \frac{3}{2}y.$$

(b) We use the tangent plane equation to get the linear approximation value :

$$f(1.1, 1.2) \approx f(1, 1) + \frac{1}{2}(1.1 - 1) + \frac{3}{2}(1.2 - 1) = 2 + 0.35 = 2.35$$

5. (14.4) The tangent plane to a surface $z = f(x, y)$ at the point $(-2, 3, 4)$ has equation $4x + 2y + z = 2$. Estimate $f(-2.1, 3.1)$.

Ans. By the given tangent equation, we get $f_x(-2, 3) = -4$ and $f_y(-2, 3) = -2$, so the linear approximation would be

$$\begin{aligned} f(x, y) &\approx f(-2, 3) + f_x(-2, 3)(x + 2) + f_y(-2, 3)(y - 3) \\ \Rightarrow f(-2.1, 3.1) &\approx 4 - 4(-2.1 + 2) - 2(3.1 - 3) = 4.2 \end{aligned}$$

6. (14.4) Approximate $(\sqrt[3]{28})^2 + (\sqrt{24})^3$.

Ans. Let $f(x, y) = (\sqrt[3]{x})^2 + (\sqrt{y})^3$. We will use the tangent plane

$$z = f(27, 25) + f_x(27, 25)(x - 27) + f_y(27, 25)(y - 25)$$

to approximate $f(28, 24)$. Note that we have chosen $(27, 25)$ as our point of tangency since $f(27, 25)$ can be easily calculated. While other points could be used, they would most certainly lead to a less accurate approximation.

$$f(27, 25) = (\sqrt[3]{27})^2 + (\sqrt{25})^3 = 3^2 + 5^3 = 9 + 125 = 134$$

$$f_x(x, y) = \frac{2}{3} \frac{1}{\sqrt[3]{x}} \quad \Rightarrow \quad f_x(27, 25) = \frac{2}{9}$$

$$f_y(x, y) = \frac{3}{2} \sqrt{y} \quad \Rightarrow \quad f_y(27, 25) = \frac{15}{2}$$

Therefore,

$$(\sqrt[3]{28})^2 + (\sqrt{24})^3 = f(28, 24) \approx 134 + \frac{2}{9}(28 - 27) + \frac{15}{2}(24 - 25) = 134 - \frac{131}{18} = 126.7\bar{2}.$$

For comparison, note that the value given by a calculator is 126.7963802.

7. (14.6) The dimensions of a closed rectangular box are measured as 20 cm, 10 cm, and 5 cm respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the volume of the box.

Ans. $V = xyz$. So,

$$dV = V_x dx + V_y dy + V_z dz = yz dx + xz dy + xy dz.$$

Since $x = 20, y = 10$ and $z = 5$, we get $dV = 50dx + 100dy + 200dz$ and since for the maximum error $dx = dy = dz = 0.2$, we get

$$dV = 50 \cdot 0.2 + 100 \cdot 0.2 + 200 \cdot 0.2 = 70.$$

8. (14.5) Let $f(x, y) = x^2 + 6y^2$.

- (a) Find the unit vector in the direction for which the directional derivative of f at the point $(-3, 4)$ is maximum.
 (b) Find the unit vectors in the directions for which the directional derivative of f at the point $(-3, 4)$ is zero.
 (c) Compute the directional derivative of f at the point $(-3, 4)$ in the direction toward the origin.

Ans. (a) Note that the gradient direction gives the maximum directional derivative. Since $\nabla f = \langle f_x, f_y \rangle = \langle 2x, 12y \rangle$, $\nabla f(-3, 4) = \langle -6, 48 \rangle$. So, the unit vector with the gradient direction would be $\frac{1}{\sqrt{65}} \langle -1, 8 \rangle$.

(b) The directional derivative in the direction of $\mathbf{u} = \langle a, b \rangle$ is $D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u}$, and at $(-3, 4)$, $D_{\mathbf{u}}f(-3, 4) = \nabla f(-3, 4) \bullet \mathbf{u} = \frac{1}{\sqrt{65}} \langle -1, 8 \rangle \bullet \langle a, b \rangle = 0$ when $a = 8/\sqrt{65}$, $b = 1/\sqrt{65}$. Hence, $\mathbf{u} = \frac{1}{\sqrt{65}} \langle 8, 1 \rangle$.

(c) If we let $P = (-3, 4)$ then the vector from P toward the origin is $\overrightarrow{PO} = \langle 3, -4 \rangle$, and we let the unit vector with that direction be $\mathbf{v} = \frac{1}{5} \langle 3, -4 \rangle$. Then,

$$D_{\mathbf{v}}f(-3, 4) = \nabla f(-3, 4) \bullet \mathbf{v} = \langle -6, 48 \rangle \bullet \frac{1}{5} \langle 3, -4 \rangle = -42$$

9. (14.5) Suppose that $f(x, y)$ is a function of two variables and that the directional derivative of f in the direction $\mathbf{u} = \langle 1, 0 \rangle$ at the point $(2, 3)$ is 5, i.e., $D_{\mathbf{u}}f(2, 3) = 5$, and that $D_{\mathbf{v}}f(2, 3) = -2$, if $\mathbf{v} = \langle 0, 1 \rangle$.

- (a) What is the gradient of f at $\langle 2, 3 \rangle$, i.e., $\nabla f(2, 3)$?
 (b) In which direction \mathbf{w} is the directional derivative $D_{\mathbf{w}}f(2, 3) = 0$?

Ans. We know that

$$D_{\mathbf{u}}f(2, 3) = \nabla f(2, 3) \bullet \mathbf{u} = 5, \quad D_{\mathbf{v}}f(2, 3) = \nabla f(2, 3) \bullet \mathbf{v} = -2.$$

If we write $\nabla f(2, 3) = \langle a, b \rangle$, the above equations simply becomes

$$\langle a, b \rangle \bullet \langle 1, 0 \rangle = a = 5, \quad \langle a, b \rangle \bullet \langle 0, 1 \rangle = b = -2$$

which gives $\nabla f(2, 3) = \langle 5, -2 \rangle$.

(b) The directional derivative $D_{\mathbf{w}}f(2, 3) = \nabla f(2, 3) \bullet \mathbf{w} = \langle 5, -2 \rangle \bullet \mathbf{w} = 0$ in the direction $\langle 2, 5 \rangle$.

10. (14.5) Let $F(x, y) = x^3 - x^2 + y^2 - y + 1$.

(a) Find the gradient $\nabla F(x, y)$.

(b) Find the directional derivative in the direction of $\langle 1, 2 \rangle$ at the point $(2, 3)$.

(c) Let $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ be a curve such that $\mathbf{r}(0) = \langle 1, 0 \rangle$ and $\mathbf{r}'(0) = \langle 1, 1 \rangle$. Let $h(t) = F(\mathbf{r}(t))$. Find $h'(0)$.

Ans. (a) $\nabla F(x, y) = \langle 3x^2 - 2x, 2y - 1 \rangle$.

(b) Unit vector in the direction $\langle 1, 2 \rangle$ is $\mathbf{u} = \frac{\langle 1, 2 \rangle}{\|\langle 1, 2 \rangle\|} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$. $\nabla F(2, 3) = \langle 8, 5 \rangle$.

The directional derivative in the direction of $\langle 1, 2 \rangle$ is

$$D_{\mathbf{u}}F(2, 3) = \nabla F(2, 3) \bullet \mathbf{u} = \langle 8, 5 \rangle \bullet \frac{1}{\sqrt{5}} \langle 1, 2 \rangle = \frac{18}{\sqrt{5}}.$$

(c) By the chain rule, $h'(t) = \nabla F(\mathbf{r}(t)) \bullet \mathbf{r}'(t)$. So,

$$h'(0) = \nabla F(1, 0) \bullet \langle 1, 1 \rangle = \langle 1, -1 \rangle \bullet \langle 1, 1 \rangle = 0.$$

11. (14.6) Evaluate $\frac{\partial h}{\partial q}$ at $(q, r) = (3, 2)$, where $h(u, v) = ue^v$, $u = q^3$, $v = qr^2$.

Ans. Note that

$$\frac{\partial h}{\partial q} = \frac{\partial h}{\partial u} \frac{\partial u}{\partial q} + \frac{\partial h}{\partial v} \frac{\partial v}{\partial q}.$$

$\frac{\partial h}{\partial u} = e^v$, $\frac{\partial h}{\partial v} = ue^v$, $\frac{\partial u}{\partial q} = 3q^2$, $\frac{\partial v}{\partial q} = r^2$. So,

$$\frac{\partial h}{\partial q} = e^v \cdot 3q^2 + ue^v \cdot r^2 = 3q^2 e^{qr^2} + q^3 r^2 e^{qr^2}.$$

At $(q, r) = (3, 2)$,

$$\left. \frac{\partial h}{\partial q} \right|_{(q,r)=(3,2)} = 27e^{12} + 27 \cdot 4e^{12} = 135e^{12}.$$

12. (14.6) Evaluate $\frac{\partial z}{\partial y}$ when $e^{xy} + \sin(xz) + y = 0$.

Ans. Take $\partial/\partial y$ on both sides of the given equation :

$$\frac{\partial}{\partial y} (e^{xy} + \sin(xz) + y) = \frac{\partial}{\partial y} (0) = 0.$$

$$\Rightarrow xe^{xy} + x \cos(xz) \cdot \frac{\partial z}{\partial y} + 1 = 0$$

Solve for $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial y} = \frac{-xe^{xy} - 1}{x \cos(xz)}.$$