1. (Cutland 9.1.7.3-4)

(a) Show that \( \{ x : \phi_x \text{ is total} \} \leq_m \{ x : W_x \text{ is infinite} \} \).

We want to translate questions about totality to questions about infinite domains. Define

\[
f(x, y) = \begin{cases} 
\phi_x(y) & \text{if } \phi_x(z) \in \mathbb{N} \text{ for all } z \leq y \\
u.d. & \text{otherwise.}
\end{cases}
\]

This function is computable by the following algorithm: on input \( x, y \), compute \( \phi_x(0), \phi_x(1), \ldots \), in turn until reach \( \phi_x(y) \) (if ever). If, for some \( i \leq y \), \( \phi_x(i) \) is undefined then the algorithm never halts and \( f(x, y) \) is undefined. Otherwise, return \( \phi_x(y) \).

Therefore, the \( s - m - n \) theorem gives a total computable function \( k(x) \) such that

\[
\phi_{k(x)}(y) \simeq f(x, y).
\]

But,

\[
\phi_x \text{ is total } \iff \phi_{k(x)} \text{ has infinite domain}
\]

since if there is some \( y \) for which \( \phi_x(y) \) is undefined, for all \( z \geq y \), \( \phi_{k(x)}(z) \) is undefined. Thus, \( k(x) \) witnesses the \( m \)-reduction.

(c) Show that neither of the above sets is \( m \)-reducible to an r.e. set.

Recall that if \( A \leq_m B \) and \( B \) is r.e. then \( A \) is r.e. (Fact 5 from lecture). Therefore, if either of the above sets were \( m \)-reducible to an r.e. set, it would itself be r.e.

However, note that each of these sets is an index set. By the Rice-Shapiro theorem, an index set is r.e. if and only if it is determined by finite approximations. Let \( \mathcal{A} = \{ \phi_e : \phi_e \text{ is total} \} \) and \( \mathcal{B} = \{ \phi_x : W_x \text{ is infinite} \} \). Any total computable function \( \theta \) is in \( \mathcal{A} \) but none of its finite approximations \( \theta \subset \theta \) are in \( \mathcal{A} \) (they are not total since they have finite domains). Thus, by the Rice-Shapiro theorem, \( \{ x : \phi_x \in \mathcal{A} \} = \{ x : \phi_x \text{ is total} \} \) is not r.e. Similarly, any total computable function is in \( \mathcal{B} \) but none of its finite approximations is in \( \mathcal{B} \). Hence, we conclude that \( \{ x : \phi_x \in \mathcal{B} \} = \{ x : W_x \text{ is infinite} \} \) is not r.e.

Since each of these sets is not r.e., neither is \( m \)-reducible to an r.e. set.

2. (Cutland 9.2.9.5) Let \( a, b \) be \( m \)-degrees.

(a) Show that the least upper bound of \( a, b \) is uniquely determined; denote it by \( a \cup b \).

Suppose we have two \( m \)-degrees \( c_1, c_2 \) that are candidates for being the least upper bound of \( a, b \). Let \( A \in a, B \in b, C_1 \in c_1, C_2 \in c_2 \). By definition of upper bound, for each \( i = 1, 2 \), \( A \leq_m C_i \) and \( B \leq_m C_i \).

But, since each \( c_i \) is a least upper bound and the above line says that each \( C_i \) is an upper bound, we have that

\[
C_1 \leq_m C_2 \quad \text{and} \quad C_2 \leq_m C_1.
\]

In other words, \( C_1 \equiv_m C_2 \), so \( c_1 = c_m \), and the least upper bound is unique.
(b) Show that if \( a \leq_m b \) then \( a \cup b = b \).

Let \( A, B \) be sets as in (b). By reflexivity, \( B \leq_m B \). And, since \( a \leq_m b \), \( A \leq_m B \). This means that \( B \) is an upper bound for \( A, B \). Since \( a \cup b \) is the least upper bound and \( A \uplus B \in a \cup b \) (proof of Theorem 9.2.8), we have \( A \uplus B \leq_m B \). But, since \( B \leq_m A \uplus B \) (also in proof of Theorem 9.2.8), we conclude \( B \equiv_m A \uplus B \) and so \( b = a \cup b \).

(c) Show that if \( a, b \) are both r.e. then so is \( a \cup b \).

Let \( A, B \) be sets as in (b) and let \( f_A, f_B \) be their (respective) partial characteristic functions. By definition of recursive enumerability, \( f_A \) and \( f_B \) are computable. It suffices to prove that \( f_{A \uplus B} \), the partial characteristic function of \( A \uplus B \), is computable.

\[
f_{A \uplus B}(x) = \begin{cases} 1 & \text{if } x \text{ is even and } f_A(x/2) = 1, \\ 1 & \text{if } x \text{ is odd and } f_B((x-1)/2) = 1, \\ \text{u.d.} & \text{otherwise}. \end{cases}
\]

Deciding if \( x \) is even or odd can be done computably, and then we use the computability of \( f_A, f_B \) to conclude that \( f_{A \uplus B} \) is computable. Thus, \( a \cup b \) contains an r.e. set and hence is r.e.

(d) Let \( A \in a \) and let \( a^* \) denote \( d_m(\overline{A}) \). Show that \( (a \cup a^*)^* = a \cup a^* \).

By the proof of Theorem 9.2.8, \( A \uplus \overline{A} \in a \cup a^* \) and \( A \uplus A \in (a \cup a^*)^* \). Hence, it suffices to prove that \( A \uplus \overline{A} \equiv_m A \uplus A \). By the definition of \( \uplus \),

\[
x \in A \uplus \overline{A} \iff \begin{cases} x \text{ is even } \land x/2 \in A, \text{or} \\ x \text{ is odd } \land (x-1)/2 \in \overline{A}. \end{cases}
\]

Similarly,

\[
x \in \overline{A} \uplus A \iff \begin{cases} x \text{ is even } \land x/2 \in \overline{A}, \text{or} \\ x \text{ is odd } \land (x-1)/2 \in A. \end{cases}
\]

Consider the (total computable) function \( f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases} \). For each even \( x \),

\[
x \in A \uplus \overline{A} \iff ((x+1)-1)/2 \in A \iff f(x) \text{ is odd and } (f(x)-1)/2 \in A \iff f(x) \in A \uplus A,
\]

\[
x \in \overline{A} \uplus A \iff ((x+1)-1)/2 \in \overline{A} \iff f(x) \text{ is odd and } (f(x)-1)/2 \in \overline{A} \iff f(x) \in A \uplus \overline{A}.
\]

Similarly, for each odd \( x \),

\[
x \in A \uplus \overline{A} \iff (x-1)/2 \in \overline{A} \iff f(x) \text{ is even and } f(x)/2 \in \overline{A} \iff f(x) \in \overline{A} \uplus A,
\]

\[
x \in \overline{A} \uplus A \iff (x-1)/2 \in A \iff f(x) \text{ is even and } f(x)/2 \in A \iff f(x) \in A \uplus \overline{A}.
\]

In other words, for all \( x, x \in A \uplus \overline{A} \iff f(x) \in \overline{A} \uplus A \) and \( x \in A \uplus \overline{A} \iff f(x) \in A \uplus \overline{A} \).

Thus the function \( f \) witnesses both that \( A \uplus \overline{A} \leq_m A \uplus \overline{A} \) and that \( A \uplus \overline{A} \leq_m A \uplus \overline{A} \).

3. (Cutland 9.4.10.5a) Prove that for any set \( A \) and (oracle) set \( B \),

\[
A \text{ is } B\text{-recursive } \iff A, \overline{A} \text{ are both } B\text{-r.e.}
\]
Suppose $A$ is $B$-recursive. Then $A \leq_T B$, so by definition there is some index $e$ such that $\phi_e^B \cong \chi_A$. Consider the functions

$$f(x) = \begin{cases} 1 & \text{if } \phi_e^B(x) = 1 \\ \text{u.d.} & \text{if } \phi_e^B(x) = 0 \end{cases}$$

$$\bar{f}(x) = \begin{cases} 1 & \text{if } \phi_e^B(x) = 0 \\ \text{u.d.} & \text{if } \phi_e^B(x) = 1. \end{cases}$$

These are both $B$-computable because they are obtained by finitely many operations (p.r. and composition) from $B$-computable functions. Moreover, they compute (respectively) the partial characteristic functions of $A, \bar{A}$. Therefore, $A, \bar{A}$ are each $B$-r.e.

Conversely, suppose that $A, \bar{A}$ are each $B$-r.e. Then, by definition, the partial characteristic functions for $A, \bar{A}$ are each $B$-computable. Consider the following algorithm to compute $\chi_A$: on input $x$, run in parallel (e.g. by alternating steps of computation) the algorithms (which may call on information about oracle $B$) which compute $f_A(x), \bar{f}_A(x)$. Exactly one of these will halt and wish to output 1. If the computation of $f_A(x)$ halts, output 1; if the computation of $\bar{f}_A(x)$ halts, output 0. By the relativised Church-Turing thesis, we have proved that $\chi_A$ is computable using oracle $B$, and hence that $A$ is $B$-recursive.

4. (Cutland 9.4.10.8) Let $A$ be any set. Show that for any set $B$,

$$B \text{ is A - r.e. } \iff B \leq_m K^A.$$  

Suppose $B$ is A-r.e. Then the partial characteristic function $f_B$ is $A$-computable. Define the function

$$f(x, y) = \begin{cases} 1 & \text{if } x \in B \\ \text{u.d.} & \text{if } x \notin B \end{cases} = f_B(x).$$

Since this function is $A$-computable, the relativised $s - m - n$ theorem says that there is a total and computable function $k(x)$ such that

$$\phi_k^A(y) \cong f(x, y).$$

But,

$$x \in B \iff f(x, k(x)) \in \mathbb{N} \iff \phi_k^A(k(x)) \in \mathbb{N} \iff k(x) \in K^A.$$ 

Thus, $k$ witnesses that $B \leq_m K^A$.

Conversely, suppose that $B \leq_m K^A$. Then there is a total computable function $g$ such that

$$x \in B \iff g(x) \in K^A \iff \phi_g^A(g(x)) \in \mathbb{N} \iff \psi_U^A(g(x), g(x)) \in \mathbb{N}.$$ 

To prove that $B$ is A-r.e, we will give an algorithm (which may ask about membership in $A$) which computes the partial characteristic function $f_B$. On input $x$, first compute $g(x)$ and then run the universal URMO program with oracle $A$ on $(g(x), g(x))$. If this computation halts, output 1. By the relativised Church-Turing thesis, we are done.

5. (Cutland 9.5.21.2) Prove that for any sets $A, B$,

$$A \leq_T B \iff K^A \leq_m K^B,$$ 

Suppose $A \leq_T B$. It suffices to prove that $K^A$ is $B$-r.e. (by the result of question 4). We first prove that $K^A$ is $A$-r.e: write the partial characteristic function of $K_A$ as

$$f_{K^A}(x) = \begin{cases} 1 & \text{if } x \in K^A \\ \text{u.d.} & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } \psi_U^A(x, x) \in \mathbb{N} \\ \text{u.d.} & \text{otherwise,} \end{cases}$$
hence an algorithm with access to oracle $A$ can compute it. Thus, $K^A$ is $A$-r.e.

But, in class we proved that for any sets $X, Y, Z$: if $X \leq_T Y$ and $Z$ is $X$-r.e. then $Z$ is $Y$-r.e. Recapping the proof: since $Z$ is $X$-r.e. there is an algorithm using oracle $X$ which computes the partial characteristic function $f_Z$. We can modify this algorithm by replacing all calls to oracle $X$ by the algorithm which computes membership in $X$ with oracle $Y$. Hence, we get an algorithm for $f_Z$ which only uses the oracle $Y$. This proves that $Z$ is $Y$-r.e.

Since we have that $K^A$ is $A$-r.e. and we assume that $A \equiv_T B$, we conclude that $K^A$ is $B$-r.e. Therefore, $K^A \leq_m K^B$.

Conversely, suppose that $K^A \leq_m K^B$. Since $A$ is $A$-r.e., the results of question 4 and transitivity give that

$$A \leq_m K^A \leq_m K^B \quad \text{hence} \quad A \leq_m K^B \quad \text{hence} \quad A \text{ is } B \text{-r.e.}$$

Similarly, $\bar{A}$ is $A$-r.e., so a similar chain of reasoning gives that $\bar{A}$ is also $B$-r.e. Applying the result of question 3, we conclude that $A$ is $B$-recursive, that is $A \equiv_T B$.

$$A \equiv_T B \iff K^A \equiv_m K^B.$$ 

This follows immediately from the first part and the definition of $\equiv_T$, $\equiv_m$:

$$A \equiv_T B \iff (A \leq_T B \land B \leq_T A) \iff (K^A \leq_m K^B \land K^B \leq_m K^A) \iff K^A \equiv_m K^B.$$