1. Write a URM program that computes the function 

\[ g(x) = \begin{cases} 
\sqrt{x} & \text{if } x \text{ is a perfect square} \\
\text{undefined} & \text{otherwise.} 
\end{cases} \]

Hint: find \( f(x, y) \) such that \( g(x) \simeq \mu y \) (\( f(x, y) = 0 \)) and use the URM program for \( f \) as a subroutine in the program for \( g \).

2. Recall the construction from lecture: given any unary function \( f \) computed by TM \( M \), there is a general recursive function \( f_m \) computing \( f \) (where \( m \) is the code for the machine \( M \)). In particular, 

\[ f_m(x) = \text{out}(m, x, \text{halt}(m, x)) \]

is defined by composition of the functions start, conf, newleft, nextstate, actn, etc. [see the handout for notation]. Explain how the construction must be changed when the original function \( f \) has arity \( k \).

Define the Ackermann function as follows:

\[ \psi(0, y) = y + 1 \quad \psi(x + 1, 0) \simeq \psi(x, 1) \]

\[ \psi(x + 1, y + 1) \simeq \psi(x, \psi(x + 1, y)) \]

3. (a) Compute \( \psi(0, 2) \), \( \psi(1, 2) \), \( \psi(2, 2) \), \( \psi(3, 2) \).

(b) Prove that \( \psi \) is well-defined total function by showing that, for all \( x, y \), \( \psi(x, y) \) can be obtained by using only a finite number of earlier values of \( \psi \).

(c) Prove that for each \( n \), the (unary) function \( f_n(y) = \psi(n, y) \) is primitive recursive.

4. This question illustrates why \( \psi \) is recursive. Define a set \( S \) of triples to be suitable if it satisfies

(i) if \( \langle x, y, z \rangle \in S \) then \( z = y + 1 \)

(ii) if \( \langle x + 1, 0, z \rangle \in S \) then \( \langle x, 1, z \rangle \in S \)

(iii) if \( \langle x + 1, y + 1, z \rangle \in S \) then there is \( u \) such that \( \langle x + 1, y, u \rangle \in S \) and \( \langle x, u, z \rangle \in S \).

(a) Prove that if \( S \) is a suitable set of triples and \( \langle x, y, z \rangle \in S \) then \( z = \psi(x, y) \).

(b) Prove that for any \( m, n \in \mathbb{N} \), the set of triples \( \langle x, y, \psi(x, y) \rangle \) used in the calculation of \( \psi(m, n) \) forms a suitable set of triples.

(BONUS) Prove the converse of (b): if \( S \) is a suitable set of triples and \( \langle x, y, z \rangle \in S \) then \( S \) contains all the earlier triples needed to calculate \( \psi(x, y) \).

Putting the pieces together: Using primitive recursive coding (see details in Cutland p. 47),

\[ A = \{ \langle x, y, v \rangle : v \text{ is the code number of a suitable set of triples and } \exists z < v(\langle x, y, z \rangle \in S_v) \} \]

has a primitive recursive characteristic function \( \chi_A \). Therefore, the function \( f(x, y) = \mu v (1 - \chi_A(\langle x, y, v \rangle) = 0) \) is general recursive and gives a code for a suitable set of triples. Moreover,

\[ \psi(x, y) = \mu z(\langle x, y, z \rangle \in S_f(x, y)). \]

Thus, \( \psi \) is general recursive.
5. This question illustrates why $\psi(x, y)$ is not primitive recursive.

(a) Prove that if $f$ is in the class of functions that can be obtained from $0, S, U^n_i$ by composition (without primitive recursion) then there is $a \in \mathbb{N}$ such that for all $x_1, \ldots, x_n$

$$f(x_1, \ldots, x_n) < \max\{x_i\} + a.$$  

(b) Show that (a) implies that the addition function is not in this class of functions.

(BONUS) Prove that if $f$ is in the class of functions obtained from the basic functions by composition and at most one use of primitive recursion then there is $c \in \mathbb{N}$ such that for all $x_1, \ldots, x_n$

$$f(x_1, \ldots, x_n) < c \max\{x_1\} + c.$$  

Explain the relevance of these observations to the fact that $\psi(x, y)$ is not primitive recursive.