1. (a) Let \( p_0 = 0, p_1 = 2, p_2 = 3, p_3 = 5, \ldots \) where \( p_i \) is the \( i \)th prime. For each \( k > 0 \), define the coding function \( c_k : \mathbb{N}^k \to \mathbb{N} \) by

\[
c_k(x_1, \ldots, x_k) = p_1^{x_1} \cdots p_k^{x_k}.
\]

Prove that each of these functions is primitive recursive.

(b) Using the Church-Turing thesis, prove that the set \( \bigcup_{k \geq 0} \mathbb{N}^k \) is effectively denumerable; that is, show that there is a computable bijection between \( \bigcup_{k \geq 0} \mathbb{N}^k \) and \( \mathbb{N} \).

2. This question proves the closure of primitive recursive functions under course-of-values recursion (also known as definition by strong induction).

Write \( \bar{x} = x_1, \ldots, x_k \). Let \( f(\bar{x}) \) and \( g(\bar{x}, y, p) \) be primitive recursive functions. Define the function

\[
h(\bar{x}, y) = \begin{cases} f(\bar{x}) & \text{if } y = 0 \\ g(\bar{x}, z, c_y(h(\bar{x}, 0), \ldots, h(\bar{x}, z))) & \text{if } y = z + 1. \end{cases}
\]

You will show that \( h \) is primitive recursive.

(a) Consider the auxiliary function

\[
\tilde{h}(\bar{x}, y) = c_{y+1}(h(\bar{x}, 0), \ldots, h(\bar{x}, y)).
\]

Prove that \( \tilde{h} \) is primitive recursive.

(b) Use \( \tilde{h} \) to prove that \( h \) is primitive recursive.

3. Let \( f \) be a recursive total function and let \( A \) be the set of all \( n \) such that the value of \( f(n) \) is different from \( f(m) \) for all \( m < n \). Prove that \( A \) is a recursive set (in the sense of Question 3 on the Midterm exam).

4. (Cutland 4.1.6, second two parts)

(a) Find the code number of the URM program

\[
T(3, 4), S(3), Z(1).
\]

(b) Find \( P_{100} \); that is, the program with code number 100.

5. (Cutland 4.2.3) Prove that every computable function has infinitely many indices.