Relevant readings: Cutland §7.1-7.4

For $A, B \subseteq \mathbb{N}$. Define the sets $A \oplus B$ and $A \otimes B$ by

$$A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$$
$$A \otimes B = \{\pi(x, y) : x \in A \text{ and } y \in B\},$$

where $\pi$ is the pairing function $\pi(x, y) = 2^x(2y + 1) - 1$.

1. (Cutland 7.1.4.1)
   (a) Prove that $A \oplus B$ is recursive if and only if $A$ and $B$ are both recursive.
   (b) Prove that if $A, B \neq \emptyset$, then $A \otimes B$ is recursive if and only if $A$ and $B$ are both recursive.
   (c) Show why the nonemptiness restriction is required in (b).

2. (Cutland 7.3.13.6)
   (a) Show that if $A$ is creative and $B \neq \emptyset$ is r.e. then $A \oplus B$ and $A \otimes B$ are creative.
   (b) Show that if $B$ is recursive then if $A \oplus B$ is creative so is $A$, and if $A \otimes B$ is creative then so is $A$.

3. (Cutland 7.3.13.9) Disjoint sets $A, B$ are said to be **effectively recursively inseparable** if there is a total computable function $f$ such that whenever $A \subseteq W_a, B \subseteq W_b$ and $W_a \cap W_b = \emptyset$ then $f(a, b) \notin W_a \cup W_b$.
   (a) Prove that if $A, B$ are effectively recursively inseparable then they are recursively inseparable.
   (b) Prove that the sets $K_0 = \{x : \phi_x(x) = 0\}$ and $K_1 = \{x : \phi_x(x) = 1\}$ are effectively recursively inseparable. **Hint:** Find a total computable function $f$ such that if $W_a \cap W_b = \emptyset$ then

   $$\phi_{f(a, b)}(x) = \begin{cases} 
   1 & \text{if } x \in W_a \\
   0 & \text{if } x \in W_b \\
   \text{u.d.} & \text{otherwise}
   \end{cases}$$

   (c) Suppose that $A, B$ are both effectively recursively inseparable. Prove that if $A, B$ are both r.e. then they are both creative.

4. (Cutland 7.4.4.1) Suppose that $A, B$ are simple sets. Show that the set $A \oplus B$ is simple.

5. (Cutland 7.4.4.3) Show that if $A$ is simple then $A \otimes \mathbb{N}$ is r.e., but is neither recursive, creative, nor simple.