1. (b) Do the same for all homomorphisms from $\mathbb{Z}_{24}$ into $\mathbb{Z}_{18}$.

Since $\mathbb{Z}_{24}$ is cyclic, a homomorphism $\phi : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ is completely determined by its value on the generator $[1]_{24}$. If $\phi([1]_{24}) = [m]_{18}$ then $\phi([x]_{24}) = x\phi([1]_{24}) = x[m]_{18} = [xm]_{18}$. By Exercise 11 of Section 2.1, $\phi$ is a function if and only if $[18] \mid 24m$ or $[3] \mid 4m$. Since $(3, 4) = 1$ this holds if and only if $3m$. Thus a complete list of all homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$ is:

$\phi_0 : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ given by $\phi_0([x]_{24}) = [0]_{18}$ for all $[x]_{24} \in \mathbb{Z}_{24}$;

$\phi_3 : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ given by $\phi_3([x]_{24}) = [3x]_{18}$ for all $[x]_{24} \in \mathbb{Z}_{24}$;

$\phi_6 : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ given by $\phi_6([x]_{24}) = [6x]_{18}$ for all $[x]_{24} \in \mathbb{Z}_{24}$;

$\phi_9 : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ given by $\phi_9([x]_{24}) = [9x]_{24}$ for all $[x]_{24} \in \mathbb{Z}_{24}$;

$\phi_{12} : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ given by $\phi_{12}([x]_{24}) = [12x]_{24}$ for all $[x]_{24} \in \mathbb{Z}_{24}$;

$\phi_{15} : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$ given by $\phi_{15}([x]_{24}) = [15x]_{24}$ for all $[x]_{24} \in \mathbb{Z}_{24}$.

19. Give an example to show that the assumption that $\phi$ is onto is needed in part (a) of Proposition 3.7.6.

Let $G_1 = S_2$ and $G_2 = S_3$. Define $\phi : G_1 \to G_2$ by $\phi(\sigma) = \sigma$ for all $\sigma \in S_2$. Then $\phi(G_1) = \{(1), (1, 2)\} = S_2$, which is a subgroup of $S_3$ but is not normal. (See Exercise 12.)

5. Let $G$ be a group with subgroup $H$. Prove that there is a one-to-one correspondence between the left and right cosets of $H$. (Your proof must include the case in which $G$ is infinite.)

Let $R = \{Ha \mid a \in G\}$ and $L = \{aH \mid a \in G\}$. Define $\phi : R \to L$ by $\phi(Ha) = a^{-1}H$. To see that $\phi$ is well-defined, suppose that $Ha = Hb$. Then $ba^{-1} \in H$ and so $(ba^{-1})^{-1} = ab^{-1} \in H$.

Thus $a^{-1}H = b^{-1}H$ and $\phi$ is well-defined. Suppose that $\phi(Ha) = \phi(Hb)$. Then $a^{-1}H = b^{-1}H$ and so $(b^{-1})^{-1}a^{-1} = ba^{-1} \in H$. Thus $Ha = Hb$, and $\phi$ is one-to-one. For any $aH \in L$ we have $\phi(Ha^{-1}) = (a^{-1})^{-1}H = aH$, which shows that $\phi$ is onto. (Remark: The obvious first choice for $\phi$, namely $\phi(Ha) = aH$ fails to be well-defined!)

11. Let $N$ be a normal subgroup of $G$. Show that the order of any coset $aN$ in $G/N$ is a divisor of $o(a)$, when $o(a)$ is finite.

Let $a \in G$, and suppose that $o(a) = m$. Then $(aN)^m = a^mN = eN = N$ and so $o(aN)|m$.

20. Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(1, 1)\rangle$ is an infinite cyclic group.

Define $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $\phi((n, m)) = n - m$. Since $\phi(n_1 + m_1, n_2 + m_2) = \phi(n_1, n_2, m_1, m_2) = (n_1 + n_2) - (m_1 + m_2) = (n_1 - m_1) + (n_2 - m_2) = \phi(n_1, m_1) + \phi(n_2, m_2)$, $\phi$ is a homomorphism of $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}$, and $\ker \phi = \langle(1, 1)\rangle$. Hence $\mathbb{Z} \times \mathbb{Z}/\langle(1, 1)\rangle \cong \mathbb{Z}$.

22. Show that $\mathbb{R}^\times/\langle-1\rangle$ is isomorphic to the group of positive real numbers under multiplication.

Define $\phi : \mathbb{R}^\times \to \mathbb{R}^\times$ by $\phi(x) = |x|$. Then $\phi(x_1x_2) = |x_1x_2| = |x_1| \cdot |x_2| = \phi(x_1)\phi(x_2)$. Given $r \in \mathbb{R}^+$, we have $r > 0$, and so $\phi(r) = |r| = r$ shows that $\phi$ is onto. Then $\ker \phi = \{x \in \mathbb{R}^\times \mid \phi(x) = 1\} = \{1, -1\} = \langle-1\rangle$. By Theorem 3.8.9, $\mathbb{R}^\times/\langle-1\rangle = \mathbb{R}^\times/\ker(\phi) \cong \phi(\mathbb{R}^\times) = \mathbb{R}^\times$.

28. Let $H$ and $N$ be normal subgroups of a group $G$, with $N \subseteq H$. Define $\phi : G/N \to G/H$ by $\phi(xN) = xH$, for all cosets $xN \in G/N$.

(a) Show that $\phi$ is a well-defined homomorphism.

Suppose that $aN = bN$. Then $b^{-1}a \in N \subseteq H$. Hence $aH = bH$ and so $\phi : G/N \to G/H$ is well-defined. Then $\phi(aNbN) = \phi(abN) = abH = aHbH = \phi(aN)\phi(bN)$. Given $aH \in G/H$, then $\phi(aN) = aH$ shows that $\phi$ is onto.

(b) Show that $(G/N)/\langle H/N \rangle \cong G/H$.

We have $\ker \phi = \{aN \in G/N \mid \phi(aN) = H\} = \{aN \in G/N \mid aH = H\} = \{aN \in G/N \mid a \in H\} = H/N$. Thus $(G/N)/\langle H/N \rangle = G/N/\ker \phi \cong G/H$ by Theorem 3.8.9.

Note that the proof is given in Theorem 7.1.2.
Addendum Solns, HW9

Throughout we use the standard fact that $\det(AB) = \det(A) \det(B)$ for $A, B \in \text{GL}_n(\mathbb{R})$, and $\det(A^{-1}) = \frac{1}{\det(A)}$ for $A \in \text{GL}_n(\mathbb{R})$.

1) Prove that $SO_2(\mathbb{R})$ is a subgroup of $O_2(\mathbb{R})$

Let $A, B \in SO_2(\mathbb{R})$. We will show $AB^{-1} = AB^T \in SO_2(\mathbb{R})$.

Note $O_2$ is a group, so $AB^{-1} = AB^T \in O_2(\mathbb{R})$.

Then $\det(AB^{-1}) = \det(A) \det(B^{-1}) = \frac{1}{\det(B)} = 1 \cdot \frac{1}{1} = 1$.

So $AB^{-1} \in O_2(\mathbb{R})$ and $\det(AB^{-1}) = 1$. Thus $AB^{-1} \in SO_2(\mathbb{R})$.

2) What is the left coset of $SO_2(\mathbb{R})$ in $O_2(\mathbb{R})$ determined by

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ?$$

Let $B = \{ A \in O_2(\mathbb{R}) \mid \det(A) = -1 \}$. We will show $B = a SO_2(\mathbb{R})$.

Let $B = a SO_2(\mathbb{R})$.

Let $b \in B$. Then $\det(ab) = \det(a) \det(b) = (-1)(-1) = 1$,

so $ab \in SO_2(\mathbb{R})$. Then $a(ab) \in a SO_2(\mathbb{R})$, but $a(ab) = (aa)b = Idb = b$,

so $b \in a SO_2(\mathbb{R})$.

$a SO_2(\mathbb{R}) \subseteq B$.

Let $c = ad \in a SO_2(\mathbb{R})$ for $d \in SO_2(\mathbb{R})$.

Then $\det(c) = \det(ad) = \det(a) \det(d) = (1)(1) = -1$, so $c \in B$.

3) Is $SO_2(\mathbb{R})$ normal in $O_2(\mathbb{R})$?

Let $d \in O_2(\mathbb{R})$. Then $\det(d) = \pm 1$, so $d \in SO_2(\mathbb{R})$ or (by pt. 2) $d \in a SO_2(\mathbb{R})$. Thus $O_2(\mathbb{R}) = SO_2(\mathbb{R}) \cup a SO_2(\mathbb{R})$, and $[O_2(\mathbb{R}) : SO_2(\mathbb{R})] = 2$.

Then (Example 3.8, 8) $SO_2(\mathbb{R})$ is normal in $O_2(\mathbb{R})$. 