The midterm exam covers the lecture material up to and including Monday, November 5. This roughly corresponds to (the sections we discussed in) Chapter 1 and 2.1-2.4 of HHM, and the corresponding parts in Prof. Verstraete's Lecture Notes Parts 1 and 4.

Some of the problems on the test may be very similar to these. On the other hand, some of these problems are harder than what will appear on the test. Solutions will not be distributed for these practice problems (other than the ones already available for homework). It is often much more productive to figure out for yourself how to start a problem and then ask for help if and when you get stuck.

**Exercise 1.** All the homework exercises: both those that you handed in and those that were recommended but not collected. Also, some examples from lectures or section were taken from the textbook. You should make sure to review them as well. That is:

- HHM 1.1.2 pp. 9-10 # 1, 2, 4, 6, 10, 11, 12, 14, 15
- HHM 1.1.3 pp. 16-17 #1, 2, 3, 4, 8, 9, 10
- Ver Part 1 p. 15 # 5
- HHM 1.2.1 pp. 20-21 # 1, 2, 5, 6
- HHM 1.2.2. p. 25 # 1, 2, 3, 8
- HHM 1.3.1 pp. 33-34 #1, 2, 3, 4
- HHM 1.3.2 pp. 37-38 #1, 2, 3, 5, 6, 10
- HHM 1.3.3 p. 42 #1, 2, 3, 5
- HHM 1.4.2 pp. 58-59 # 1,2, 5, 7
- HHM 2.1 pp. 134-137 # 1, 5, 6, 7, 11, 12
- HHM 2.2 pp. 142-144 # 2, 3, 4, 5, 6, 7

**Exercise 2.** List all (labelled) graphs of order 0, 1, 2, 3, 4 whose vertices are subsets of \(\{1,2,3,4\}\).

List all the isomorphism types of graphs of these orders.

**Exercise 3.** Draw graphs with the following degree sequences, or prove that no such graphs exist.

- (a) 2, 3, 3, 3, 3, 3 (order = 6)
- (b) 0, 1, 2, 3, 4 (order = 5)
- (c) 1, 1, 2, 2 (order = 4)
- (d) 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 (order = 10)

**Exercise 4.** (a) Draw a graph with vertices representing the numbers 1, 2, \ldots, 10, in which two vertices are connected by an edge if and only if one is a divisor of the other.

(b) Draw a graph with vertices representing the numbers 1, 2, \ldots, 10, in which two vertices are connected by an edge if and only if they have no common divisor larger than 1.

(c) Find the number of edges and the degrees in these graphs, and check their relationship.

(d) Are the graphs from (a) and (b) complements of one another? Why or why not?

**Exercise 5.** Consider \(G = (V, E)\) defined as follows. Let \(V\) be the set of all odd prime numbers less than 20. Let \(E = \{(a, b) \in V \times V : a, b \text{ both prime, and differ by } 2\}\). Find the (maximal connected) component of 7. How many components does \(G\) have? List them.
Exercise 6.  (a) Show (by an example) that there is a connected graph $G$ and $e \in E(G)$ such that $G - e$ is not connected.

(b) Prove that for any connected graph $G$, if $e \in E(G)$ belongs to a cycle in $G$ then $G - e$ is connected.

Exercise 7. Prove that a vertex $v$ is a vertex of the connected component of $G$ containing vertex $u$ if and only if $G$ contains a path connecting $u$ and $v$.

Exercise 8. Prove that no edge of $G$ can connect vertices in different connected components.

Exercise 9. Prove that at least one of $G$, $\bar{G}$ is connected.

Exercise 10. Prove that connecting two vertices $u$ and $v$ in a graph $G$ by a new edge creates a new cycle if and only if $u$ and $v$ are in the same connected component of $G$.

Exercise 11. In HW 2 (1.1.2 #6), you proved that every closed odd walk in a graph contains an odd cycle.

Is it true that every closed even walk in a graph contains an even cycle? Prove or give a counterexample.

Exercise 12. List all isomorphism types of trees of order 2, 3, 4, 5. If the vertices are given labels, how many trees are there of each isomorphism type?

Exercise 13 (Ver Section 4.3). For any graph $G$, we define a block of $G$ to be a maximal connected subgraph with no cut vertex. We define a new graph, $G_B$ such that

\[ V(G_B) = \{ v \in V(G) : v \text{ is a cut vertex of } G \} \cup \{ v_B : B \subseteq V(G) \text{ is a block in } G \} \]

and

\[ xy \in E(G_B) \iff (x \in y \text{ and } x \text{ is a cut vertex in } G, y \text{ is a block of } G) . \]

(a) Prove that every block is either $K_2$ or a graph which contains a cycle.

(b) Prove that $G_B$ is a tree for any choice of a connected graph $G$.

For parts (c), (d), (e), let $G$ be the following graph.

(c) List all the blocks of $G$.

(d) Draw the tree $G_B$.

(e) Draw a spanning tree of $G$.

(f) Is $G_B$ a spanning tree of $G$? Why or why not?

(g) Characterize the graphs $G$ where $G_B$ is a spanning tree of $G$.

Exercise 14. Consider $S$, the ordinary English alphabet (with 26 letters).
(a) How many 12-element subsets of $S$ do not contain the letter $Q$?
(b) How many 12 element subsets of $S$ contain the letter $J$?

Exercise 15. Three people each select a main dish from a menu of six items.

(a) How many choices are possible (i) if we record who selected which dish (as the waiter should), and (ii) if we ignore who selected which dish (as the chef could)?
(b) Suppose it’s late in the evening and only two servings are left of each dish. What are the answers to (i) and (ii) in this case?

Exercise 16. An exam consists entirely of three True-False questions. Out of a class of thirty students, two students have identical exam papers (they attempted the same questions and answered in the same way on the questions they attempted). Is this similarity compelling evidence of cheating?

Exercise 17 (Ver Part 1 # 12). Prove that the number of sequences of 1s and 2s of length $k$ which add up to $n$ is

$${n \choose k}$$

Exercise 18. An unscrupulous stockbroker recommends the purchase or sale of 10 highly speculative stocks. He buys 1024 postcards, which he sends to 1024 prospective clients, and he gives a different recommendation on each card.

(a) Assume that after six months, all ten stocks will have changed in value (either up or down). Argue that 1024 is exactly the number of cards the stockbroker needs to ensure that some prospective client will have a recommendation that is right on all 10 stocks.
(b) How many prospective clients receive a card with at least eight out of the ten recommendations proving to be correct?

Exercise 19 (Ver Part 1 # 10). Prove the following identity by induction on $n$:

$$\sum_{k=1}^{n} k^3 = \left( \frac{n+1}{2} \right)^2 .$$

Exercise 20 (Ver Part 1 #11). Prove the following inequality by induction on $n \geq 1$:

$$\binom{2n}{n} < 4^n .$$

Exercise 21 (Ver Part 1 # 6). Let $A$ be the set of all sequences of positive integers of length $k$ which add up to $n$, and let $B$ be the set of all subsets of $\{1, \ldots, n-1\}$ of size $k-1$. Find a bijection from $A$ to $B$ and use it to deduce that $|A| = \binom{n-1}{k-1}$ for $n \geq k \geq 1$.

Exercise 22. Prove the following identity by giving a bijection between two relevant sets. For $n \geq 3$:

$$\binom{n}{k} = \binom{n-3}{k} + 3 \binom{n-3}{k-1} + 3 \binom{n-3}{k-2} + \binom{n-3}{k-3} .$$

Exercise 23. How many (labelled) induced subgraphs are there of a graph $G$ of order $n$?