The final exam covers all the lecture material. This roughly corresponds to (the sections we discussed in) Sections 1.1-1.4, 1.6, and 2.1-2.6 of HHM, and the corresponding parts in Prof. Verstraete’s Lecture Notes Parts 1-4.

Some of the problems on the test may be very similar to these. On the other hand, some of these problems are harder than what will appear on the test. Solutions will not be distributed for these practice problems (other than the ones already available for homework). It is often much more productive to figure out for yourself how to start a problem and then ask for help if and when you get stuck.

**Exercise 1.** All the homework exercises: both those that you handed in and those that were recommended but not collected. Also, some examples from lectures or section were taken from the textbook. You should make sure to review them as well. That is:

- HHM 1.1.2 pp. 9-10 # 1, 2, 4, 6, 10, 11, 12, 14, 15
- HHM 1.1.3 pp. 16-17 #1, 2, 3, 4, 8, 9, 10
- Ver Part 1 p. 15 # 5
- HHM 1.2.1 pp. 20-21 # 1, 2, 5, 6
- HHM 1.2.2. p. 25 # 1, 2, 3, 8
- HHM 1.3.1 pp. 33-34 #1, 2, 3, 4
- HHM 1.3.2 pp. 37-38 #1, 2, 3, 5, 6, 10
- HHM 1.3.3 p. 42 #1, 2, 3, 5
- HHM 1.4.2 pp. 58-59 # 1, 2, 5, 7
- HHM 2.1 pp. 134-137 # 1, 5, 6, 7, 11, 12
- HHM 2.2 pp. 142-144 # 2, 3, 4, 5, 6, 7
- HHM 2.4 pp. 154-156 # 2,3,5,13
- HHM 2.5 pp. 161-162 # 1,2,3,4,5,10
- Ver Part 1 pp. 16 #1, 3, 7, 8
- HHM 1.6.1 pp. 87-88 #1,2,3
- HHM 2.6 pp. 165-166 # 2,3
- HHM 2.6.2 pp. 170-171 # 5,6,7,8,9
- HHM 2.6.3 pp. 175-176 # 9,10
- HHM 2.6.5 pp. 183-184 # 2, 4,6,7
- Ver Part 2 pp. 33-34 #1, 6, 7
- Ver Part 3 pp. 10-11 # 3, 5.

Also, all the practice questions from the midterm.

*Note: Since you have the practice questions from the midterm, the following new exercises are more heavily weighted towards the second half of the course. However, the final exam will be cumulative and will be representative of the entire course.*

**Exercise 2.** Consider the graph
(a) Draw all subgraphs of this graph. How many distinct isomorphism types are represented?
(b) Draw all induced subgraphs of this graph. How many distinct isomorphism are represented?

Exercise 3. Which of $K_n, C_n, P_n, S_n$ are bipartite graphs?

Exercise 4. Let $G$ be a graph of order $n$ and size $m$. What is the minimum number of connected components it may have? What is the maximum number of connected components it may have?

Exercise 5 (Uses Linear Algebra). Recall that if $A$ is an $n \times n$ matrix, $\lambda$ is an eigenvalue of of the eigenvector $x_1, \ldots, x_n$ of $A$ if and only if

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Prove that if $A$ is the adjacency matrix of a $d$-regular graph of order $n$, then $d$ is an eigenvalue of $A$ for the eigenvector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Exercise 6. A vertex $v \in V(G)$ is call pivotal for a pair of vertices $x, y \in V(G)$ if it lies on every shortest path between $x$ and $y$ (and isn’t equal to $x$ or $y$).

(a) Give an example of a graph in which every vertex is pivotal for at least one pair of vertices.
(b) Give an example of a graph of order at least 4 in which there is a single vertex $v$ that is pivotal for every pair of vertices (which doesn’t include $v$ itself).
(c) For each of your examples, compute the radius and diameter of the graph.

Exercise 7 (HHM 1.3.2 # 7,8).

(a) Prove that every edge in a tree is a bridge.
(b) Prove that every nonleaf in a tree is a cut vertex.

Exercise 8. Consider the following graph.

(a) Find a spanning tree for this graph.
(b) Assign weights to the edges of this graph in such a way that there are (at least) two minimum weight spanning trees. Find these trees.
(c) Assign weights to the edges of this graph in such a way that there is a unique minimum weight spanning trees. Find this tree.

Exercise 9. Let $T$ be a tree and $S \subseteq V(T)$ be such that for each $u, v \in S$ there is a path in $T$ with endpoints $u$ and $v$ and all of whose vertices are in $S$. Prove that the subgraph of $T$ induced by $S$ is a tree. Bonus: Prove that the path existence condition is necessary by finding a tree $T$ and $S \subseteq V(T)$ such that the subgraph of $T$ induced by $S$ is not a tree.
Exercise 10. For $G$ a graph and $e \in E(G)$, define $G/e$ to be the graph where $e$ is removed and then its endpoints are treated as a single vertex (and any duplicate edges are removed). Birkhoff’s Theorem (HHM 1.49 p. 98) says: If $G$ is a graph and $e \in E(G)$, then
\[ c_G(k) = c_{G-e}(k) - c_{G/e}(k). \]
Prove this theorem. Then, use Birkhoff’s Theorem to prove the following facts for any graph $G$ of order $n$:
- $c_G(x)$ is a polynomial in $x$ of degree $n$.
- The leading coefficient of $c_G(x)$ is 1.
- The constant term of $c_G(x)$ is 0.
- The coefficients of $c_G(x)$ alternate in sign.
- The size of $G$ is the absolute value of the coefficient of $x^{n-1}$.

Hint: Proceed by induction on the size of $G$.

Exercise 11. Which of the following numbers can be the size of the union of a set with 5 elements and a set with 9 elements: 4, 6, 9, 10, 14, 20?

Exercise 12. Let $A$ be a finite set. Give a bijection between the odd-sized subsets of $A$ and the even-sized subsets of $A$.

Hint: break into cases depending on whether $A$ has even or odd size.

Exercise 13. Give a bijection between the following two sets $A$ and $B$, thereby proving that they have the same size. What is this size?

\[ A = \{ \langle a_1, \ldots, a_n \rangle : a_i \in \{0, 1\} \text{ for } 1 \leq i \leq n \} \]

\[ B = \{ \text{all finite sequences (of any length) whose entries are positive integers and sum to } n+1 \} \]

Recall that a bijection is a one-to-one and onto map.

Exercise 14. Use the inclusion-exclusion principle to prove that the number of surjections (onto functions) from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ is given by
\[ n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m. \]

Deduce that
\[ n^n - \binom{n}{1}(n-1)^n + \binom{n}{2}(n-2)^n - \cdots + (-1)^{n-1}\binom{n}{n-1}1^n = n! \]

Exercise 15.

(a) We have 20 different presents that we want to distribute to 12 children. It is not required that every child get something; it could even happen that we give all the presents to the same child. In how many ways can we distribute the presents?

(b) We have 20 kinds of presents; this time, we have a large supply of each kind. We want to give presents to 12 children. Again, it is not required that every child gets something; but no child can get two copies of the same present. In how many ways can we give presents?

Exercise 16. Find the values of $\binom{n}{k}$ for $k = 0, 1, n-1, n$ algebraically. Explain these results in terms of the combinatorial meaning of $\binom{n}{k}$.
Exercise 17. Prove the identity
\[
\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}
\]
first algebraically, and then by finding a bijection between two sets of relevant sizes.

Exercise 18. (a) List all subsets of \(\{a, b, c, d, e\}\) containing \(\{a, e\}\) but not containing \(c\).
(b) Consider a set \(A\) of size \(n\), a subset \(B \subseteq A\) of size \(k \leq n\), and an element \(x \in A \setminus B\). Find a formula for the number of subsets of \(A\) which contain all the elements in \(B\) but do not contain \(x\). Is your answer consistent with (a)?

Exercise 19. Starting from Washington, DC, how many ways can you visit five of the 50 state capitals and then return to Washington? Assume that there are direct flights between any two state capitals.

Exercise 20. You want to send postcards to 12 friends. In the shop, there are only 3 kinds of postcards. In how many ways can you send the postcards, if:

(a) there is a large number of each kind of postcard, and you want to send one card to each friend.
(b) the shop has only 4 of each kind of postcard, and you want to send one card to each friend.
(c) What generating function encodes the answer to each of the above problems.

Exercise 21. In a group of people, 18 like to play chess, 23 like to play soccer, 21 like biking, and 17 like hiking. The number of those who like to play both chess and soccer is 9. Also, 7 people like both chess and biking, 6 people like both chess and hiking, 12 like soccer and biking, 9 like soccer and hiking, and 12 like biking and hiking. There are 4 people who like chess, soccer, and biking, 3 who like chess, soccer, and hiking, 5 who like chess, biking, and hiking; and 7 who like soccer, biking and hiking. Finally, there are 3 people who like all four activities. In addition, we know that everyone likes at least one of these activities. How many people are there in the group?

Exercise 22. A drawer contains 6 pairs of black, 5 pairs of white, 5 pairs of red, and 4 pairs of green socks.

(a) How many single socks do we have to take out to make sure that we take out two socks with the same color?
(b) How many single socks do we have to take out to make sure that we take out two socks with different colors?

Exercise 23. Prove the following identity:
\[
\binom{n}{0} \binom{m}{k} + \binom{n}{1} \binom{m}{k-1} + \cdots + \binom{n}{k-1} \binom{m}{1} + \binom{n}{k} \binom{m}{0} = \binom{n+m}{k}
\]

Exercise 24. Consider the identity
\[
\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.
\]
(a) Prove the identity by induction on \(k \geq 0\).
(b) Prove the identity combinatorially by arguing that to choose a \((k+1)\)-element set from \(\{1, 2, \ldots, n+k+1\}\), you can first choose the largest element, and then the rest.
Exercise 25. Suppose you want to choose a \((2k + 1)\)-element subset of \(\{1, \ldots, n\}\). You decide to do this by choosing first the middle element, then the \(k\) elements to its left, then the \(k\) elements to its right. Formulate the combinatorial identity you get from this. **Bonus:** Can you prove this identity by algebraic means?

Exercise 26. Prove the following identities:

(a) \[ \sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n+1} \]

(b) \[ \sum_{k=0}^{m} \binom{n-k}{m-k} = \binom{n+1}{m} \]

Exercise 27. Let \(a_{n,k}\) be the number of compositions of \(n\) with \(k\) parts where each part is greater than or equal to 4. Let \(a_n\) be the number of compositions of \(n\) with any number of parts, where each part is greater than or equal to 4.

(a) For fixed \(k\), determine a generating function, \(G_k(x)\), that encodes this sequence of numbers \((a_{n,k})\). Then, use formal power series methods to give a formula for \(a_{n,k}\).

(b) Determine a generating function, \(G(x)\), that encodes the sequence of numbers \(a_n\). Then, use formal power series methods to give a formula for \(a_n\).

(c) Confirm that your answers are consistent for \(n = 10\). That is, check that the formulas you derived satisfy

\[ a_{10} = \sum_{k=0}^{10} a_{10,k}. \]

Exercise 28. Repeat the above exercise but for compositions of \(n\) where each entry must be in the set \(\{1, 2, 3\}\).

Exercise 29. In the lecture notes (Ver), on p. 31, several different combinatorial problems are listed (all of which are counted by the Catalan numbers). Prove that these problems are counted by the same sequence of numbers by exhibiting bijections between each pair of problems or by showing that they satisfy the same recurrence equation as the Catalan numbers:

\[ C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} \quad C_0 = 1. \]

- **Ballot Problem** The ballot problem is the number of possible processes of \(2n\) votes for candidates \(A\) and \(B\) such that at any stage in the voting process, candidate \(A\) has at least as many votes as candidate \(B\).

- **Path Problem.** This problem consists in determine the number of paths \((x_1, x_2, \ldots, x_{2n})\) in the plane such that \(x_1 = (0, 0)\), and for any \(i < 2n\), \(x_{i+1} = x_i + (1, 1)\) or \(x_{i+1} = x_i + (1, -1)\), and the path never dips below the \(x\)-axis. In other words, at any step, we move one step up and one step right, or one step down and one step right, and we never go below \(y = 0\).

- **Binary Bracketing Problem.** Suppose we are given a list \(x_1, x_2, x_3, \ldots, x_{2n}\) of letters. In how many ways can we place \(2n\) brackets \(\text{“} \) or \(\text{“} \text{“}\) between these letters in a sensible mathematical way? In particular, the number of right brackets (reading from left to right) should never exceed the number of left brackets, and the total number of left and right brackets is equal.

- **Handshake Problem.** If \(2n\) people are sitting around a table, and each person shakes hands with exactly one other person, how many ways can people shake hands if no people have to cross arms to shake hands?