Let $G$ be a graph and let $v \in V(G)$. Define the component of $G$ generated by $v$ to be the subgraph of $G$ induced by the set

$$\{x \in V(G) : \text{there is an } x-v \text{ path in } G\}.$$ 

(1) Prove that the component of $G$ generated by $v$ is a maximal connected subgraph of $G$.

(2) Prove that $v$ is in the component of $G$ generated by itself.

(3) Let $v, u \in V(G)$. Prove that if $u$ is in the component of $G$ generated by $v$ then $v$ is in the component of $G$ generated by $u$.

(4) Let $v, u, w \in V(G)$. Prove that if $u$ is in the component of $G$ generated by $v$ and if $v$ is in the component of $G$ generated by $w$ then $u$ is in the component of $G$ generated by $w$. 
(1) Let $S \subseteq V(G)$ be the vertex set of the component of $G$ generated by $v$. To prove that the subgraph induced by $S$ is a maximal connected subgraph, we need to show that if $x \in V(G) \setminus S$, then the subgraph induced by $S \cup \{x\}$ is not connected. Towards a contradiction, suppose that this subgraph is connected. By definition, this means that there is a path between any two vertices. In particular, there is a path between $v$ and $x$. But, then $x \in S$ by definition of the component generated by $v$. This contradicts our assumption that $x \notin S$.

(2) The length 0 path with one vertex, $v$, witnesses that there is a path from $v$ to $v$ in $G$ and that, therefore, $v$ is in the component of $G$ generated by itself.

(3) If $u$ is in the component of $G$ generated by $v$ then there is a sequence of vertices $x_1, \ldots, x_k$ such that $u, x_1, \ldots, x_k, v$ is a path in $G$. But, paths are non-directional. So $v, x_k, \ldots, x_1, u$ is also a path in $G$ and so $v$ is in the component of $G$ generated by $u$.

(4) If $u$ is in the component of $G$ generated by $v$ then there is a path $u, x_1, \ldots, x_k, v$. If $v$ is in the component of $G$ generated by $w$ then there is a path $v, y_1, \ldots, y_j, w$. Concatenating these two paths gives the walk $u, x_1, \ldots, x_k, v, y_1, \ldots, y_j, w$. Earlier, we proved that if $G$ contains a walk between two vertices then there is a path in $G$ with those endpoints. Therefore, $u$ is in the component of $G$ generated by $w$. 