Consider the identity
\[
\binom{n}{2} + \binom{n+1}{2} = n^2.
\]

(1) Prove the identity using the algebraic formula for binomial coefficients.

(2) Prove the identity by interpreting each side as the size of a set and giving a bijection between these sets.
(1) We compute
\[
\binom{n}{2} + \binom{n+1}{2} = \frac{n!}{2(n-2)!} + \frac{(n+1)!}{2(n-1)!} = \frac{n!(n-1) + (n+1)!}{2(n-1)!}
\]
\[
= \frac{(n-1)!((n-1) + (n+1))}{2(n-1)!} = \frac{1}{2} (n^2 - n + n^2 + n) = n^2.
\]

(2) The RHS is the number of sequences of length 2 from an \( n \)-element set. Applying the sum rule for the LHS, we partition this set into two pieces, \( A, B \) such that \( |A| + |B| = \{|\langle a, b \rangle : a, b \in \{1, \ldots, n\}\} \). If we can prove that \( |A| = \binom{n}{2} \) and \( |B| = \binom{n+1}{2} \), we are done. Let
\[
A = \{\langle a, b \rangle : a, b \in \{1, \ldots, n\}, a < b\}, \quad B = \{\langle a, b \rangle : a, b \in \{1, \ldots, n\}, a \geq b\}.
\]

To compute \( |A| \), note that since we only consider ordered pairs with different elements and we fix the ordering, the pair \( \langle a, b \rangle \) is specified by choosing a two-element subset of \( \{1, \ldots, n\} \), which is \( \binom{n}{2} \). To count the number of elements in \( B \), we rewrite its elements as
\[
B = \{\langle a, b \rangle : 1 \leq b \leq a \leq n\} = \{\langle a' - 1, b \rangle : 1 \leq b \leq a' - 1 \leq n\}
\]
\[
= \{\langle a' - 1, b \rangle : 1 \leq b < a' \leq n + 1\}.
\]

Thus, each pair \( \langle a, b \rangle \) in \( B \) is specified by choosing a two-element subset of \( \{1, \ldots, n+1\} \) and using its smaller element as \( b \) and one-less-than-its bigger element as \( a \). There are \( \binom{n+1}{2} \) many such subsets. Thus, \( |A| + |B| = \binom{n}{2} + \binom{n+1}{2} \) and we are done.