Assigned questions to hand in:

(1) Let $T$ be a tree of order $n > 1$. Show that the number of leaves is

$$2 + \sum_{v \in V(T) : \deg_T(v_i) \geq 3} (\deg_T(v_i) - 2).$$

_HHM 1.3.2.10 p. 38_

Solution: We proceed by induction on the order of $n$. *Notation: we will use* $\deg_T(v)$ *to denote the degree of vertex* $v$ *with respect to the graph* $T$.

- **Base case:** Suppose $|V(T)| = 2$. Then both vertices are leaves. The equation we are considering has an empty summation (because no vertex has degree greater than or equal to 3) and correctly gives the answer, 2.

- **Induction step:** For the induction assumption, we assume that the number of leaves in each tree with order at most $n$ agrees with the equation. Suppose $T$ is a tree with $|V(T)| = n + 1$. Since $T$ has at least two leaves (Theorem 1.14 of HHM), let $v$ be one of these leaves and consider $T - v$. This is still a tree and so the induction hypothesis holds for $T - v$. That is, the number of leaves in $T - v$ is

$$2 + \sum_{v \in V(T - v) : \deg_{T - v}(v_i) \geq 3} (\deg_{T - v}(v_i) - 2).$$

Since $v$ is a leaf of $T$, it has one neighbor in $T$, call it $u$. Since the neighbors of $u$ in $T$ are $v$ plus the neighbors of $u$ in $T - v$,

$$\deg_T(u) = 1 + \deg_{T - v}(u).$$

Also, note that for each vertex $w \in V(T) \setminus \{u, v\}$, $\deg_T(w) = \deg_{T - v}(w)$. We consider two cases.

- **Case 1:** $u$ is a leaf of $T - v$. In this case, the number of leaves in $T - v$ is the same as the number of leaves of $T$. By the above observation, $\deg_T(u) = 1 + \deg_{T - v}(u) = 2$. Therefore, the set of vertices in $V$ with degree greater than or equal 3 is the same as the set of vertices in $T - v$ with degree greater than or equal to 3. Moreover,

$$2 + \sum_{v_i \in V(T) : \deg_T(v_i) \geq 3} (\deg_T(v_i) - 2) = 2 + \sum_{v_i \in V(T - v) : \deg_{T - v}(v_i) \geq 3} (\deg_{T - v}(v_i) - 2)$$

_ind. hyp._ number of leaves in $T - v$

_case assump._ number of leaves in $T$,

as required.
– Case 2: $u$ is not a leaf of $T - v$. In this case, the number of leaves of $T - v$ is one less than the number of leaves of $T$. Also, $\deg_T(u) \geq 3$ and we have
\[
2 + \sum_{v_i \in V(T): \deg_T(v_i) \geq 3} (\deg_T(v_i) - 2) = 2 + (\deg_T(u) - 2) + \sum_{v_i \in V(T): \deg_T(v_i) \geq 3, v_i \neq u} (\deg_T(v_i) - 2)
\]
\[
= 2 + (1 + \deg_{T-v}(u) - 2) + \sum_{v_i \in V(T-v): \deg_{T-v}(v_i) \geq 3, v_i \neq u} (\deg_{T-v}(v_i) - 2)
\]
\[
= 1 + 2 + \sum_{v_i \in V(T-v): \deg_{T-v}(v_i) \geq 3} (\deg_T(v_i) - 2)
\]

\text{Ind. Hyp.} \quad 1 + \text{number of leaves in } T - v
\]
\text{Case assump.} \quad \text{number of leaves in } T,
\]
as required.

(2) Using Kruskal’s algorithm, find a minimum weight spanning tree of the graphs below (Figure 1.44 in HHM). In each case, determine (and justify with proof) whether the minimum weight spanning tree is unique.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{two_weighted_graphs.png}
\caption{Two weighted graphs.}
\end{figure}

\textit{HHM 1.3.3.5 p. 42}

\textbf{Solution:} The edges highlighted in yellow are marked by Kruksal’s algorithm for the minimum weight spanning tree. Namely, for the graph on the left, all edges with weight 2, 3, 5 are marked. For the graph on the right, the edges with weight 1, 2, 4, 5, 6, 7, 8, 10 are marked.

Each of these minimum weight subtrees is unique. To prove this, we use the following two lemmas. (Their proofs are left as an exercise.)

\textbf{Lemma (1).} Let $G$ be a graph containing a cycle $v_1, \ldots, v_k$. If there is an edge $e$ in this cycle with weight strictly larger than all other edges, then $e \notin E(T)$ for any minimum weight spanning tree $T$.

\textbf{Lemma (2).} If the weight function is one-to-one (that is, no weight is repeated) then there is a unique minimum weight spanning tree.
Lemma (2) immediately gives the uniqueness of the minimum weight spanning tree of the right-hand graph. To apply Lemma (1) to the graph on the left-hand-side, we observe that each of the unmarked edges in the graph have maximum weight in some cycle in the graph and thus cannot be included in any minimum weight spanning tree by the lemma.

(3) A ballot lists ten candidates for city council, eight candidates for the school board, and five bond issues. The ballot instructs voters to choose up to four people running for city council, rank up to three candidates for the school board, and approve or reject each bond issue. How many different ballots may be cast, if partially completed (or empty) ballots are allowed?

*HHM 2.1 p. 136 #7*

**Solution:** The choices for city council, school board, and bond issues are independent so we apply the product rule. For city council, each ballot represents an unordered choice of 0, 1, 2, 3 or 4 people from a 10 person set. Thus, there are

\[
\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} = 386
\]

many ways to fill (or partially fill) that part of the ballot. For the school board, a response is a sequence of length at most 3 from an 8 person set, with no repetitions. So, there are

\[
\binom{8}{0} + \binom{8}{1} + 2\binom{8}{2} + 3!\binom{8}{3} = 401
\]

choices. Finally, each bond issue may be accepted, rejected, or left blank. Thus, this part of the ballot constitutes 5 independent choices from a three element set: there are \(3^5 = 243\) many of these. The number of different ballots that may be cast is therefore

\[
386 \cdot 401 \cdot 243 = 37612998.
\]

(4) Suppose a positive integer \(N\) factors as \(N = p_1^{n_1} \cdots p_m^{n_m}\), where \(p_1, \ldots, p_m\) are distinct prime numbers and \(n_1, \ldots, n_m\) are all positive integers. How many different positive integers are divisors of \(N\)?

*HHM 2.1 p. 137 #11*

**Solution:** Each divisor of \(N\) is uniquely determined by its prime factors and their powers. That is, by the sequence \((i_1, \ldots, i_m)\), where \(0 \leq i_j \leq n_j\) for each \(j \in \{1, \ldots, m\}\). Thus, we count the number of such sequences:

\[
(n_1 + 1) \cdots (n_j + 1).
\]

(5) Use algebraic methods to prove the cancellation identity: If \(n\) and \(k\) are nonnegative integers and \(m\) is an integer with \(m \leq n\) then

\[
\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.
\]

This identity is very useful when the left side appears in a sum over \(k\), since the right side has only a single occurrence of \(k\).

*HHM 2.2 p. 142 #3*
Solution: We use the expansion identity for binomial coefficients. On the LHS we have:
\[
\binom{n}{k} \binom{k}{m} = \frac{n!}{k!(n-k)!} \frac{k!}{m!(k-m)!} = \frac{n!}{m!(n-k)!(k-m)!}.
\]
On the RHS, we get:
\[
\binom{n}{m} \binom{n-m}{k-m} = \frac{n!}{m!(n-m)!} \frac{(n-m)!}{(k-m)!(n-m-(k-m))!} = \frac{n!}{m!(k-m)!(n-k)!}.
\]
Since these are equal, we are done.