109 Spring 2011 - Practice Problems for Final Exam

Some of the problems on the test may be very similar to these. On the other hand, some of these problems are harder than what will appear on the test. Solutions will **not** be distributed for these practice problems (other than the ones already available for homework). It is often much more productive to figure out for yourself how to **start** a problem and then ask for help if and when you get stuck.

The final exam is cumulative. The textbook exercises below reflect those assigned or mentioned throughout the whole course. However, to save paper, I have not duplicated the non-textbook exercises that were given for practice for the midterm. So, the non-textbook exercises below only represent material from the lectures after the midterm.

Exercise. All the homework exercises: both those that you handed in and those that were recommended but not collected. Also, some examples from lectures or section were taken from the textbook. You should make sure to review them as well. That is:

- Chapter 1: 1, 2
- Chapter 2: 1, 3, 5
- Chapter 3: 1, 2, 3, 6, 8i
- Chapter 4: 2, 3, 5, 7
- Chapter 5: 1, 2, 4
- Part I Problems: 1, 2, 3, 4, 5, 7, 9, 10, 11, 13.
- Chapter 6: 1, 2, 5, 6
- Chapter 7: 1, 2(i, iii, v), 4 (iii, iv), 7
- Chapter 8: 1, 3, 4
- Chapter 9: 1 (i, ii), 4, 5,
- Chapter 10: 1, 3
- Chapter 11: 1, 2, 5 (i,ii)
- Part II Problems: 2, 4, 10, 11, 12, 14, 16, 17, 18
- Limits and continuity homework and section work.
- Chapter 12: 2,4,6
- Chapter 14: 1, 2, 4
- Part III Problems: 2, 5,6,7, 11, 28
- Chapter 15: 1i, ii, 2, 3
- Chapter 16: 1
- Chapter 17: 1, 2
- Chapter 18: 1
- Part IV Problems: 7, 13, 14, 15i
- Chapter 19: 2, 4ii
- Chapter 20: 1i, ii, iii, iv, 2i
- Chapter 21: 3, 5i
- Chapter 22: 1i, ii, vii, viii, 2, 3
- Chapter 23: 1, 2

Exercise. Consider the statement

If $x, y, z \in \mathbb{Z}$ and xz divides yz then x divides y.

(a) Someone tries to convince you the statement is false by the following example:

$$x = 4, y = 6, z = 16.$$

Are you convinced?

(b) Someone tries to convince you the statement is true by the following proof:

Suppose $x, y, z \in \mathbb{Z}$ such that $xz | yz$ but $x \not| y$. Then $y|x$ so there is integer c such that $x = cy$. Thus, $xz = cyz = c(yz)$ so $yz|xz$, and this is a contradiction because $xz|yz$.

Explain at least one thing that is wrong with this proof.

(c) Is the statement true or false? Prove.

Exercise. Let $f : A \to B$ be a function and consider the following statements.

(i) For all $a \in A$, there exists a $b \in B$ such that $f(a) = b$.
(ii) For all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.
(iii) There exists an $a \in A$ such that for all $b \in B$, $f(a) = b$.
(iv) There exists a $b \in B$ such that for all $a \in A$, $f(a) = b$.

Answer the following questions.

(a) Which of the above statements are always true (i.e. for any function f)?
(b) Which of the above statements are always false?
(c) Which of the above statements are sometimes true? For each such statement give an example of one function f for which it is true.
(d) Which of the above statements are sometimes false? For each such statement give an example of one function f for which it is false.

Exercise. Let $g : A \to B$ and $f : B \to C$ be functions.

(a) Prove that if $f \circ g$ is surjective then f is surjective.
(b) Prove that the converse is not true.

Exercise. State the definition of a set being countable. Prove that every subset of a countable set is countable.

Exercise. Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

Exercise. Prove that the product of three consecutive natural numbers is divisible by 6.

Exercise. Let S be a set with 17 elements. How many subsets of S have an even number of elements?

Exercise. Let $n \in \mathbb{Z}^+$ be the number of people at a party (assume $n > 2$). Prove that there are at least 2 people who know the same number of people at the party. Assume that if A knows B then B knows A.

Exercise. Consider the relation on $\mathbb{R}$ defined by

$$x \sim y \qquad \iff \qquad x^2 - y^2 = 0.$$

(a) Prove that this is an equivalence relation.
(b) List the members of [3].

Exercise. Consider the set $S = \{a, b, c, d\}$ and let $R \subseteq S \times S$ be the relation on S given by

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, a), (c, d), (d, c)\}$$

Is R reflexive? Is R symmetric? is R transitive? Is R an equivalence relation?

Exercise. Find $a, b, \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ such that $a^2 \equiv b^2 \mod m$ but $a \not\equiv b \mod m$.

Exercise. For each of the following values of a, b find (a, b) and find $m, n \in \mathbb{Z}$ such that $ma + nb = (a, b)$.

(a) $a = 901$, $b = 952$
(b) $a = 219$, $b = 69$

(c) $a = 377$, $b = 233$.

Exercise (V.4)**.** This exercise proves that the trick for checking divisibility by 9 of a number by adding up the digits and checking if the sum is divisible by 9 works.

(a) Prove that for all $n \in Z^+$, $10^n \equiv 1 \mod 9$.
(b) Suppose that a positive integer is written in decimal notation as $n = a_k a_{k-1} \cdots a_2 a_1 a_0$ where $0 \leq a_i \leq 0$. Prove that n is divisible by 9 if and only if the sum of its digits $a_k + a_{k-1} + \cdots + a_1 + a_0$ is divisible by 9.
(c) Is 33129513 divisible by 9?

Exercise (V.7)**.** What is the last digit of 2^{1000}?

Exercise (V.10,11)**.** Solve the linear congruence

$$234x \equiv 16 \mod 366.$$

Use the result to solve the diophantine equation

$$234x + 366y = 36.$$

Exercise. Which of the following diophantine equations has no integers solutions x, y?

(a) $7x + 3y = 1$
(b) $10x = 7y = 723$
(c) $9x + 33y = 6$
(d) $21x + 35y = 3$
(e) $x + y = 0$
(f) $17x - 22y = 1$.

Exercise. Compute the greatest common divisor of 64 and 200 in two ways:

(a) Use the Euclidean algorithm.
(b) Write 64 and 200 each as a product of primes.

Exercise.

(a) Write formally the following statement: There exists a prime number p for which there exists an integer x such that $x^3 + x^2 + x = p - 1$.
(b) Prove or disprove the statement in (a) above.

Exercise. For all $n \in \mathbb{Z}^+$ prove that 7 divides $6^n + 1$ if and only if n is odd. *Hint: do* **not** *use induction.*

Exercise.

(a) Prove that for any $n \in \mathbb{Z}$, either n^2 is divisible by 4 or $n^2 \equiv 1 \mod 4$.
(b) If $x, y, z, r \in \mathbb{Z}^+$ and r is odd, prove that if $x^2 + y^2 + z^2 = r^2$ then exactly one of x, y, z is odd.

Exercise. Prove that for all $n \in \mathbb{Z}^+$, one of $n, n+4, n+8, n+12, n+16$ is a multiple of 5.