109 Spring 2011 - Supplement on continuity of real-valued functions.

Earlier, we defined functions in general (Eccles chapter 8). An important special case consists of functions on the reals, $f : \mathbb{R} \to \mathbb{R}$.

Example.

- Polynomials, exponentials, trigonometric functions
- Modulus function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- Step function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

- Floor and ceiling functions

$$\lfloor x \rfloor = \text{largest integer less than } x \qquad \lceil x \rceil = \text{smallest integer greater than } x$$

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in X$. Then f has a **limit** L at x_0 if for each $\epsilon \in \mathbb{R}^+$, there is a $\delta \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}$,

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon.$$

Example. The function given by $f(x) = x^2$ has a limit 0 at $x_0 = 0$.

Proof. Let $\epsilon \in \mathbb{R}^+$. We want to find δ such that for each x with $0 < |x| < \delta$, $0 < |x^2| < \epsilon$. Define $\delta = \sqrt{\epsilon}$. Then suppose $0 < |x| < \delta$.

$$|x^2| = x^2 < \delta^2 = (\sqrt{\epsilon})^2 = \epsilon.$$

□

Example. The modulus function has a limit 0 at $x_0 = 0$.

Proof. Let $\epsilon \in \mathbb{R}^+$. We want to find δ such that for each x with $0 < |x| < \delta$, $0 < ||x| - 0| < \epsilon$. Define $\delta = \epsilon$. Then suppose $0 < |x| < \delta$.

$$||x| - 0| = |x| < \delta = \epsilon.$$

□

Example. The function given by $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ does not have a limit at 0.

Proof. Notice that if $x < 0$ then $f(x) = -1$ and if $x > 0$ then $f(x) = 1$. We will show that for any $L \in \mathbb{R}$, L is not the limit of f at $x_0 = 0$. Let L be any number and let $\epsilon = 1$.

- Case 1: $L \geq 0$. Suppose δ is a positive number. Pick some negative number x such that $0 < |x| < \delta$. For this x, $f(x) = -1$ so

$$|f(x) - L| = |-1 - L| = |-(L + 1)| = L + 1 \geq \epsilon.$$

- Case 2: $L < 0$. Suppose δ is a positive number. Pick some positive number $0 < x < \delta$. Then $f(x) = 1$ and

$$|f(x) - L| = |1 - L| \geq 1 = \epsilon.$$

Thus, in either case, there is some x in the δ-neighbourhood of x_0 whose function value is too far away from L. □

Definition. Let $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$. Then f is **continuous** at x_0 if for each $\epsilon \in \mathbb{R}^+$, there is a $\delta \in \mathbb{R}^+$ such that if $x \in \mathbb{R}$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. In symbols:

$$\forall \epsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall x \in \mathbb{R} (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon).$$

We say that f is **continuous** if it is continuous at each point in its domain.

Example. Prove that the modulus function is continuous at 0.

Proof. Let $\epsilon \in \mathbb{R}^+$. Define $\delta = \epsilon$. Then, if $x \in \mathbb{R}$ and $|x - 0| < \delta$, we are in one of two cases:

- Case 1: $x \geq 0$. Then $0 < x < \delta$. In this case,
  $$|f(x) - f(0)| = |x - 0| = |x| < \delta = \epsilon,$$
  as required.
- Case 2: $x < 0$. Then $-\delta < x < 0$. In this case,
  $$|f(x) - f(0)| = |-x - 0| = |-x| = |x| < \delta = \epsilon,$$
  as required again. □

Example. Prove that the step function is continuous at $x_0 = 1$ but is not continuous at $x_0 = 0$.

Proof. To prove that $H(x)$ is continuous at $x_0 = 1$, we need to prove that

$$\forall \epsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall x \in \mathbb{R} (|x - 1| < \delta \implies |H(x) - 1| < \epsilon).$$

Suppose $\epsilon \in \mathbb{R}^+$ is given. Define $\delta = \frac{1}{2}$ (notice that in this case, our choice of δ doesn't depend on ϵ). Then for each $x \in \mathbb{R}$, if

$$|x - 1| < \frac{1}{2} \qquad \text{then} \qquad \frac{1}{2} < x < \frac{3}{2}$$

so x is guaranteed to be positive. Therefore, $H(x) = 1$. That is,

$$|H(x) - 1| = |1 - 1| = 0 < \epsilon.$$

The second part asks us to prove that $H(x)$ is not continuous at $x_0 = 0$. So, we need to prove that

$$\exists \epsilon \in \mathbb{R}^+ \forall \delta \in \mathbb{R}^+ \exists x \in \mathbb{R} (|x - 0| < \delta \text{ and } |H(x) - 1| \geq \epsilon)$$

We get to choose ϵ, so choose $\epsilon = \frac{1}{2}$. Given $\delta > 0$, let $x = \frac{-\delta}{2}$.

Why? We want x to be very close to 0 but negative because $H(x)$ acts differently on negative numbers from how it behaves at $x_0 = 0$.

Then

$$|x - 0| = \left| \frac{-\delta}{2} - 0 \right| = \frac{\delta}{2} < \delta$$

and

$$|H(x) - 1| = |0 - 1| = 1 > \frac{1}{2} = \epsilon.$$

□

The following theorem is often stated without proof in calculus classes. We now have all the ingredients to prove it. (Note: however, that the implication (b) $\implies$ (c) is a little tricky.)

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. Then TFAE ("the following are equivalent")*

(a) *f is continuous at x_0.*
(b) *If $\langle x_n \rangle$ is a sequence in $\mathbb{R}$ that converges to x_0 then the sequence $\langle f(x_n) \rangle$ converges to $f(x_0)$.*
(c) *f has a limit at x_0 and $\lim_{x\to x_0} f(x) = f(x_0)$.*

Proof. Prove (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).

(a) $\implies$ (b) Suppose f is continuous at x_0 and $\langle x_n \rangle$ is a sequence that converges to x_0. Let $\epsilon > 0$. Since f is continuous at x_0, there is $\delta > 0$ such that

$$|x - x_0| < \delta \qquad \implies \qquad |f(x) - f(x_0)| < \epsilon.$$

But $\lim_{n\to\infty} x_n = x_0$ so there is N such that for all $n \geq N$

$$|x_n - x_0| < \delta.$$

Thus, for this N, for all $n \geq N$,

$$|f(x_n) - f(x_0)| < \epsilon.$$

(b) $\implies$ (c) We prove the contrapositive. Namely, we will prove that if

there is $\epsilon > 0$ such that for all $\delta > 0$, there is some x such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \epsilon$

then

there is a sequence which converges to x_0 but whose image sequence does not converge to $f(x_0)$.

So assume there is $\epsilon_0 > 0$ such that for all $\delta > 0$, there is some x such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \epsilon$. In particular, we consider $\delta_1 = \frac{1}{2}, \delta_2 = \frac{1}{4}, \delta_3 = \frac{1}{8}, \ldots$, and in general, $\delta_n = \frac{1}{2^n}$. Since each $\delta_n > 0$, the assumption guarantees that there is some number, call it x_n, such that

$$|x_n - x_0| < \delta_n = \frac{1}{2^n} \qquad \text{and} \qquad |f(x_n) - f(x_0)| \geq \epsilon_0.$$

Now, consider the sequence $\langle x_n \rangle$. First, we prove that $\lim_{n\to\infty} x_n = x_0$. To do so, we need to show that

$$\forall \epsilon \in \mathbb{R}^+ \exists N \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+ (n \geq N \implies |x_n - x_0| < \epsilon).$$

Given some $\epsilon \in \mathbb{R}^+$, define $N = \lceil -\log_2(\epsilon) \rceil + 1$. Then, if $n \geq N$

$$|x_n - x_0| \leq 2^{-n} \leq 2^{-N} < 2^{\log_2(\epsilon)} = \epsilon.$$

But, we also prove that $\lim_{n\to\infty} f(x_n) \neq f(x_0)$. We need to show that

$$\exists \epsilon \in \mathbb{R}^+ \forall N \in \mathbb{Z}^+ \exists n \in \mathbb{Z}^+ (n \geq N \text{ and } |f(x_n) - f(x_0)| \geq \epsilon).$$

The witness will be $\epsilon = \epsilon_0$ from the beginning of this proof because we defined it to be such that **all** $f(x_n)$ are at least ϵ_0 away from $f(x_0)$. Formally, if $N \in \mathbb{Z}^+$, let $n = N$ and notice that

$$n \geq N \qquad \text{and} \qquad |f(x_n) - f(x_0)| \geq \epsilon_0.$$

Thus, we have shown that it is not the case that if $\langle x_n \rangle$ is a sequence in $\mathbb{R}$ that converges to x_0 then the sequence $\langle f(x_n) \rangle$ converges to $f(x_0)$. In other words, the proof of the contrapositive is complete.

(c) $\implies$ (a) Suppose $\lim_{x \to x_0} f(x) = f(x_0)$. Let $\epsilon > 0$. By definition of limit of a function, there is $\delta > 0$ such that

$$0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

This is part of the requirement in the definition of continuity; it remains to consider the case $|x - x_0| = 0$. In this case, $x = x_0$ and $f(x) - f(x_0) = 0$.

$\square$

Note: the above can be made to work with functions whose domains are subsets of $\mathbb{R}$ *but then need to worry about accumulation points.*