

MATH 109: Intro to Mathematical Reasoning

Midterm Exam May 6, 2011

Name: _____

PID: _____

Section: _____

INSTRUCTIONS — READ THIS NOW

- This test has 5 problems on 7 pages worth a total of 50 points. Look over your test package **right now**. If you find any missing pages or problems please ask for another test booklet.
- Write your name and your section number **right now**. Please have your ID available for the proctor to check during the exam.
- Show your work. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly.
- Turn off and put away all cellphones, calculators, and other electronic devices.
- This exam is 50 minutes long.
- Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty.

Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Signature of Student



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1. _____

2. _____

3. _____

4. _____

5. _____

Total: _____

1. (10 points) Determine whether each of the following is **True**, **False**, or **neither**. You need not justify your answers for this question.

(a) For all propositions P and Q ,

P or (not Q) is equivalent to $(\text{not } P) \implies Q$.

Answer:

(b) $\{x \in \mathbb{Z}^+ : 1 \leq x \leq 3\}$

Answer:

(c) $|\{x \in \mathbb{Z}^+ : 1 \leq x \leq 3\}| > 2$

Answer:

(d) $\forall x \in \mathbb{R} \exists y \in \mathbb{R} (x + y = 1)$

Answer:

(e) The sequence $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ defined by $f(n) = n^2$ is a null sequence (has limit equal to 0).

Answer:

2. (10 points) Consider the functions $f : \mathbb{N}_2 \rightarrow \mathbb{N}_3$ and $g : \mathbb{N}_3 \rightarrow \mathbb{N}_2$ defined by

$$f(x) = 2x - 1 \quad \text{and} \quad g(x) = \text{the smallest integer bigger than or equal to } \frac{x}{2}.$$

(a) List the elements of the graph of f .

$$\{(1, 1), (2, 3)\}$$

(b) What is the domain of $g \circ f$?

$$\mathbb{N}_2$$

(c) For $x = 2$, compute

★ $f(x)$

Answer:

★ $g(x)$

Answer:

★ $g(f(x))$

Answer:

3. (10 points) Prove by induction on n that, for all positive integers n , $4^n - 1$ is divisible by 3.

- **Base case** ($n = 1$). Want to show that $4^1 - 1$ is divisible by three.

But, $4^1 - 1 = 3 = 3 \cdot 1$, so there is an integer (namely 1) such that 3 times it equals $4^1 - 1$. Thus, the definition for divisibility by 3 is satisfied.

- **Inductive step** Suppose $k \in \mathbb{Z}^+$ and $4^k - 1$ is divisible by 3. We want to show that $4^{k+1} - 1$ is also divisible by 3.

$$4^{k+1} - 1 = 4(4^k) - 1 = 4(4^k - 1 + 1) - 1 = 4(4^k - 1) + 3.$$

By the inductive hypothesis, there is some $q \in \mathbb{Z}$ such that $4^k - 1 = 3q$. Thus, we can write

$$4^{k+1} - 1 = 4(3q) + 3 = 3(4q + 1).$$

Since $4q + 1 \in \mathbb{Z}$, we have found the witness to $4^{k+1} - 1$ being divisible by 3.

4. (10 points) Consider the statement

For finite sets X and Y , if there is a function $f : X \rightarrow Y$ that is a surjection then $|X| \geq |Y|$.

- (a) To prove **this statement** by **contradiction** what are you allowed to assume, and what do you need to prove?

Assume:

- X and Y are (arbitrary) finite sets.
- there is some surjection $f : X \rightarrow Y$.
- $|X| < |Y|$.

Want to prove:

Some contradiction. That is, prove (P and ‘not P ’) for some proposition P .

- (d) To prove **the converse** of this statement by a **direct** proof what are you allowed to assume, and what do you need to prove?

Assume:

- X and Y are (arbitrary) finite sets.
- $|X| \geq |Y|$.

Want to prove:

There is a surjection from X to Y .

5. (10 points)

(a) (6 points) Consider the statement

$$A \cap (A \cup B) \subseteq A$$

(where A and B are sets). Translate this sentence to one that only contains the membership operation \in , the sentence connectives ('not', 'and', 'or', \implies , \iff) and quantifiers.

$$\forall x ((x \in A) \text{ and } (x \in A \text{ or } x \in B)) \implies x \in A$$

(b) (4 points) Consider the statement

$$(A - B) \cup C = A - (B \cup C)$$

(i) Find examples of sets A, B, C which make the statement true.

Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$ and $C = \emptyset$.

Then $(A - B) \cup C = \{1, 3\} \cup \emptyset = \{1, 3\}$ and $A - (B \cup C) = \{1, 2, 3\} - \{2, 4, 6\} = \{1, 3\}$.

(ii) Find examples of sets A, B, C which make the statement false.

Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$ and $C = \{5, 7, 9\}$.

Then $(A - B) \cup C = \{1, 3\} \cup \{5, 7, 9\} = \{1, 3, 5, 7, 9\}$ but $A - (B \cup C) = \{1, 2, 3\} - \{2, 4, 6, 5, 7, 9\} = \{1, 3\}$.