109 Spring 2011 - Counting

Exercise. A true-false test has 10 questions.

(a) In how many ways can a student answer the questions if every question is answered?

Some subset of the 10 questions will get the answer "true" and the rest will be answered "false". Therefore, it is sufficient to count the number of subsets of the 10 questions. Or, equivalently, the number of function $f : \mathbb{N}_{10} \to \{\text{"true"}, \text{"false"}\}$. This is $2^{10} = 1024$.

(b) In how many ways can a student answer the questions if there is a penalty for guessing and the student leaves some (or all) answers blank?

In this case, we are counting the number of functions $f : \mathbb{N}_{10} \to \{\text{"true"}, \text{"false"}, \text{"blank"}\}$. This is $3^{10} = 59049$.

Exercise. Prove that if $n \in \mathbb{Z}^+$ then $\binom{2n}{n}$ is even.

Proof. Recall the recursive formula for the binomial coefficients:

$$\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n-1}{n-1}.$$

But, $(2n-1) - n = n - 1$, and recall that $\binom{k}{r} = \binom{k}{k-r}$ for all $k \in \mathbb{Z}^+$ and $0 \leq r \leq k$. Therefore,

$$\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n-1}{n-1} = 2\binom{2n-1}{n-1}$$

and since $\binom{2n-1}{n-1} \in \mathbb{Z}$, we are done. $\square$

Exercise. Prove that $\mathbb{R}$ is an infinite set.

Proof. Suppose for a contradiction that $\mathbb{R}$ is finite. By definition, that means it is either the empty set or has a finite cardinality. Since $1 \in \mathbb{R}$, $\mathbb{R} \neq \emptyset$. Therefore, there is $n \in \mathbb{Z}^+$ such that $|\mathbb{R}| = n$. Equivalently, there is a function $f : \mathbb{N}_n \to \mathbb{R}$ with is a bijection. Since bijections have inverses, there is a function $f^{-1} : \mathbb{R} \to \mathbb{N}_n$ which is a bijection, hence an injection. Notice that $\mathbb{N}_{n+17} \subset \mathbb{R}$. Therefore, we can restrict f to $\mathbb{N}_{n+17}$ to get a function $f \restriction \mathbb{N}_{n+17} : \mathbb{N}_{n+17} \to \mathbb{N}_n$ which is an injection (since the restriction of an injection is an injection). Lemma 10.1.4 says that if there is an injection from $\mathbb{N}_m$ to $\mathbb{N}_n$ then $m \leq n$. In our case, $m = n + 17$ and we conclude that $n + 17 \leq n$, a contradiction. $\square$

Note: in the above, the number 17 *was chosen arbitrarily; any positive number would work.*