109 Spring 2011 - Division Algorithm

Exercise. Find the greatest common divisor of 2047 and 1633 and find integers m and n such that

$$460 = 2047m + 1633n.$$

We use the Euclidean algorithm:

$$\begin{aligned} 2047 &= 1633(1) + 414 \\ 1633 &= 414(3) + 391 \\ 414 &= 391(1) + 23 \\ 391 &= 23(17) + 0. \end{aligned}$$

Therefore

$$(2047, 1633) = (1633, 414) = (414, 391) = (391, 23) = 23.$$

Reading the computation backwards gives us 23 as an integral linear combination of 2047 and 1633:

$$\begin{aligned} 23 &= 414 - 391(1) \\ &= 414 - [1633 - 414(3)] = 414(4) + 1633(-1) \\ &= [2047 - 1633](4) + 1633(-1) = 2047(4) + 1633(-5). \end{aligned}$$

Notice that $460 = 23 \cdot 20$. Therefore,

$$460 = 20\big[2047(4) + 1633(-5)\big] = 2047(80) + 1633(-100).$$

Hence, $m = 80$ and $n = -100$.

Exercise (IV.7)**.** Recall the definition of the Fibonacci sequence (5.4.2):

$$u_1 = 1 \qquad u_2 = 1 \qquad u_{k+1} = u_k + u_{k-1} \text{ for all } k \geq 1$$

Prove that for all $n \in \mathbb{Z}^+$, $\gcd(u_{n+1}, u_n) = 1$.

Proof. We proceed by induction on n.

Base case (n=1) Then we want to show that $(u_1, u_2) = 1$. By definition of the Fibonacci sequence, $u_1 = u_2 = 1$. Moreover, for any $d \in \mathbb{Z}^+$, $(d, d) = d$ since $d|d$. Therefore, $(1, 1) = 1$ and we are done.

Inductive step Suppose that $k \in \mathbb{Z}^+$ and $(u_{k+1}, u_k) = 1$. We want to show that $(u_{k+2}, u_{k+1}) = 1$.

$$u_{k+2} = u_{k+1} + u_k = u_{k+1}(1) + u_k.$$

Therefore, by Theorem 16.1.2 (with $a = u_{k+2}$, $b = u_{k+1}$ and $r = u_k$),

$$(u_{k+2}, u_{k+1}) = (u_{k+1}, u_k).$$

Moreover, by the inductive hypothesis, this is 1.

□