# 109 Spring 2011 - Quantifiers and Functions - Solutions

**Exercise** (II.11)**.** Give a proof or a counterexample for each of the following statements.

(i) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y > 0)$
True: given $x \in \mathbb{R}$ define $y = 2 - x$.

(ii) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x - y > 0)$
True: given $x \in \mathbb{R}$ define $y = 2 + x$.

(iii) $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(x + y > 0)$
False. We prove that the negation, $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y \leq 0)$, is true. Given $x$, let $y = -x$.

(iv) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(xy > 0)$
False. Counterexample: $x = 0$.

(v) $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(xy > 0)$
False. We prove the negation, $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(xy \leq 0)$. Given $x$, let $y = 0$.

(vi) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(xy \geq 0)$
True: given $x \in \mathbb{R}$ let $y = 0$.

(vii) $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(xy \geq 0)$
True: choose $x = 0$.

(viii) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y > 0 \text{ or } x + y = 0)$
True: given $x \in \mathbb{R}$ define $y = 2 - x$.

(ix) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y > 0 \text{ and } x + y = 0)$
False: the inner predicate can never be made true.

(x) $\big(\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y > 0)\big)$ and $\big(\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y = 0)\big)$
True: for first conjunct, given $x$ define $y = 2 - x$; for the second conjunct, given $x$ define $y = -x$.

**Exercise** (II.14)**.** Define functions $f$ and $g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and $g(x) = x^2 - 1$. Find the functions $f \circ f$, $f \circ g$, $g \circ f$, and $g \circ g$. List the elements of the set $\{x \in \mathbb{R} : fg(x) = gf(x)\}$.

$$\begin{gathered} f \circ f(x) = (x^2)^2 = x^4 \\ (f \circ g)(x) = f(g(x)) = f(x^2 - 1) = (x^2 - 1)^2 = x^4 - 2x^2 + 1 \\ (g \circ f)(x) = g(f(x)) = g(x^2) = (x^2)^2 - 1 = x^4 - 1 \\ g \circ g(x) = (x^2 - 1)^2 - 1 = (x^4 - 2x^2 + 1) - 1 = x^4 - 2x^2 \end{gathered}$$

A number is in $\{x \in \mathbb{R} : fg(x) = gf(x)\}$ if and only if

$$x^4 - 2x^2 + 1 = x^4 - 1.$$

That is, if and only if

$$-2x^2 = -2 \qquad \text{i.e.} \qquad x^2 = 1.$$

Thus, this set is $\{-1, 1\}$.